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THEORY OF GROUPS

Miller, G. A. Groups containing a maximal proper subgroup of order 4. Proc. Nat. Acad. Sci. U. S. A. 27, 445–448 (1941). [MF 5165]

Apart from groups of order eight, and groups which are direct products of a group of order four and a group of odd prime order p, the only groups having maximal subgroups of order four are (1) dihedral or dicyclic groups of order $4p$, and (2) groups G of order $4p$ (with $p=4k+1$) or of order $4p^2$ (with p odd) containing an invariant subgroup with respect to which the quotient group is cyclic of order 4.

J. S. Frame (Providence, R. I.).

Miller, G. A. Maximal invariant proper subgroups of a finite group. Proc. Nat. Acad. Sci. U. S. A. 27, 587–590 (1941). [MF 5910]

The author discusses the maximal proper subgroups of a group. If all are of the same order, the group is a prime power group. If all are of different orders the group is cyclic. If a group has at most two maximal proper subgroups of the same order, then all these subgroups have different orders. In an historical comment at the end, the author refers to an early use of the term "Abelian group" by H. Weber. *J. S. Frame* (Providence, R. I.).

Jabber, M. A. Determination of the groups of order 180. Bull. Calcutta Math. Soc. 33, 57–70 (1941). [MF 6170]

The author determines and gives generating relations for the 37 groups of order 180, of which one is the direct product of the icosahedral group with a cyclic group of order three, and all the others contain a single subgroup of order 5. The method used depends on Schreier's extension theorem referred to the invariant subgroup of order 5, and the 14 possible quotient groups of order 36. The relation between the Schreier method and the method of Senior and Lunn for the determination of all the soluble groups of a given finite order is discussed in a concluding paragraph.

J. S. Frame (Providence, R. I.).

Fubini, Guido. Some properties of finite discontinuous groups. Publ. Inst. Mat. Univ. Nac. Litoral 3, 49–64 (1941). (Spanish. English summary) [MF 5605]

The author determines all the finite groups which contain exactly three greatest cyclic subgroups (that is, cyclic subgroups which are not part of cyclic subgroups of higher order). He then proceeds to an investigation of a "generalization of the Abelian groups"; he fails to notice, however, that the class of groups under consideration is exactly the well-known class of Abelian and Hamiltonian groups.

R. Baer (Urbana, Ill.).

Zappa, Guido. Sui gruppi risolubili d'ordine dispari. Ann. Scuola Norm. Super. Pisa (2) 9, 147–161 (1940). [MF 5459]

Applying Ph. Hall's theory of regular p-groups [Proc. London Math. Soc. (2) 36, 29–95 (1934)] and his characterization of soluble groups [J. London Math. Soc. 3, 98 (1928)] and using a solubility criterion of W. K. Turkin [Math. Ann. 111, 281–284 (1935)] as well as one of his own estimates for the number of elements of a given order in a simple group [Rend. Sem. Mat. Univ. Roma (4) 2, 173–176 (1938)], the author proves that a group is soluble if its order has either of the following forms: $p_1^a p_2^2 \cdots p_r^2 p_{r+1} \cdots p_{r+n}$ with $a \leqq p_1$, $m+r+1 \leqq p_1$ or $p_1^a p_2^b p_3^2 \cdots p_r^2 p_{r+1} \cdots p_{r+n}$ with $p_1 < p_i$ $(i \neq 1)$, $b \leqq p_2$, $m+r \leqq p_1$. *R. Baer* (Urbana, Ill.).

Zappa, Guido. Sulla risolubilità di taluni gruppi finiti. Boll. Un. Mat. Ital. (2) 3, 19–27 (1940). [MF 5571]

Proof of the theorem that a finite group is soluble if its order is of the form $\prod_{i=1}^{r} p_i^{a_i}$ with $a_i \leqq 3$ and $r < p_i$ [cf. the preceding review]. *R. Baer* (Urbana, Ill.).

Kingston, J. M. A generalization of the plane lattice groups. Trans. Roy. Soc. Canada. Sect. III. (3) 35, 27–40 (1941). [MF 6022]

This paper is mainly concerned with the groups

(1) $A^l = B^m = C^n = ABC = D$, $(AC^{-n/l})^b (A^{-1}C^{n/l})^c = D^r = 1$,

where $l^{-1}+m^{-1}+n^{-1}=1$ and $m \leqq l \leqq n$. [For the case when $r=1$, see Burnside, Theory of Groups, Cambridge, England, 1911, pp. 413–417.] The author shows that the value of r is unrestricted, and that the order of such a group is r times that of the corresponding Burnside group. When r is prime to lmn, we have the stronger statement that (1) is the direct product of the corresponding Burnside group and the cyclic group of order r. For, choosing integers a, b, c such that $al \equiv bm \equiv cn \equiv 1 \pmod{r}$, we can write $A = A_0 D^a$, $B = B_0 D^b$, $C = C_0 D^c$, where A_0, B_0, C_0 are elements of (1) which satisfy

$$A_0^l = B_0^m = C_0^n = A_0 B_0 C_0 = (A_0 C_0^{-n/l})^b (A_0^{-1} C_0^{n/l})^c = 1.$$

H. S. M. Coxeter (Toronto, Ont.).

Iwasawa, Kenkiti. Über die endlichen Gruppen und die Verbände ihrer Untergruppen. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 171–199 (1941). [MF 5634]

It is the object of this investigation to translate properties of the partially ordered set of all the subgroups of a finite group into group-theoretical properties. Some instances of this problem have been discussed by O. Ore [Duke Math. J. 4, 247–269 (1938); 5, 431–460 (1939)]. Of the author's numerous new results the following seem most interesting. The set of subgroups of the finite group G is a finite projective geometry whose lines carry at least three points if, and only if, either G is an Abelian group all of whose elements $(\neq 1)$ are of the same order p; or else G contains an Abelian normal subgroup H with the following properties: (i) every element $(\neq 1)$ in H has the same order p; (ii) G/H is a cyclic group of order a prime number q; (iii) to every element g in G that is not in H there

exists an integer r such that $ghg^{-1}=h^r$ for every h in H, $r\not\equiv 1 \bmod p$, $r^q\equiv 1 \bmod p$. [For another characterization of this class of groups, cf. R. Baer, Amer. J. Math. **61**, 1–44 (1939), Theorem 11.2.] The group G is termed modular if $A(B\cap C)=AB\cap C$ whenever A, B, C are subgroups of G such that $C\leqq A$. Any two subgroups R, S of the finite group G satisfy $RS=SR$ if, and only if, G is the direct product of modular p-groups. The p-group G is modular if, and only if, G is either Hamiltonian or else there exists an Abelian normal subgroup H of G with the following properties: (a) G/H is cyclic; (b) to every element g which generates G modulo H there exists an integer $s>0$ (which is at least 2, if $p=2$) such that $ghg^{-1}=h^{1+p^s}$ for every h in H. Every finite modular group is the direct product of modular p-groups and of groups G of the following type (and all these groups are modular): G is of order p^aq^b (p, q primes), possesses an Abelian normal subgroup S of order p^a all of whose elements $\neq 1$ are of order p, a cyclic subgroup (t) of order q^b, and there exists an integer r such that $tst^{-1}=s^r$ for every s in S, $r\not\equiv 1 \bmod p$, $r^q\equiv 1 \bmod p$. *R. Baer.*

Clifford, A. H. and Mac Lane, Saunders. Factor-sets of a group in its abstract unit group. Trans. Amer. Math. Soc. **50**, 385–406 (1941). [MF 5718]

Let Γ be a finite group of order n. The free Abelian group $\mathfrak{H}$ of rank $n-1$ generated by n symbols H^σ (σ ranging over Γ), subject to the single condition that their product is 1, is the abstract unit group of Γ. If τ is in Γ, the mapping $H^\sigma\to H^{\sigma\tau}$ generates an automorphism of $\mathfrak{H}$, and in this way there is obtained a representation of Γ by automorphisms of $\mathfrak{H}$. The problem considered in this paper had its origin in a study of normal algebraic number fields by MacLane and Schilling [Trans. Amer. Math. Soc. **50**, 295–384 (1941); cf. these Rev. **3**, 102]; it is that of determining, from the point of view of Schreier's theory, all extensions $\mathfrak{G}$ of $\mathfrak{H}$ by Γ which realize the above representation. Since two extensions are equivalent if and only if the factor-sets of Γ in $\mathfrak{H}$ determined by them are associate, the problem reduces at once to a study of factor-sets of Γ in $\mathfrak{H}$. All such factor-sets form an Abelian group $F\mathfrak{H}$, of which the identity element 1 is the factor-set, each element of which is the identity element of $\mathfrak{H}$. The factor-sets associate to 1 form a subgroup $T\mathfrak{H}$ of $F\mathfrak{H}$ and of central interest is the group $M\mathfrak{H}=F\mathfrak{H}/T\mathfrak{H}$; in particular, the order of $M\mathfrak{H}$ is the number of non-equivalent extensions of $\mathfrak{H}$ by Γ. Now let Ω be any algebraically closed field of characteristic not dividing the order n of Γ. Schur [J. Reine Angew. Math. **127**, 20–50 (1904)] has introduced the group $\mathfrak{M}$ of classes of associate factor-sets of Γ in Ω, this group being independent of Ω. The authors conjecture that $M\mathfrak{H}$ and $\mathfrak{M}$ are isomorphic, and the main result of the paper is a proof of this conjecture for the case in which Γ is solvable. Extensive use is made of certain functions called crossed characters of Γ in $\mathfrak{H}$, from the analogy with a crossed representation of Γ by matrices [see Speiser, Math. Z. **5**, 1–6 (1919); Schur, ibid. **5**, 7–10 (1919)]. Some of the results concerning crossed characters are of interest in themselves, in addition to their use in establishing the main result. *N. H. McCoy.*

Kurosch, A. Ringtheoretische Probleme, die mit dem Burnsideschen Problem über periodische Gruppen in Zusammenhang stehen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 233–240 (1941). (Russian. German summary) [MF 5174]

A fundamental problem in the study of periodic groups, that is, groups each of whose elements are of finite period, is the problem of Burnside: (G) Is every periodic group with a finite number of generators a finite group? The author studies certain closely related problems (R), (K) for a ring R and a field K. Problems (R) (and (K)) may be stated: Is a ring (field) which is algebraic over P and has a finite number of generators over P, a finite algebra (division algebra) over P? This is shown to be true if all elements of (R) ((K)) are of degree not greater than 3 over P, and also if K has only two generators and its elements are of degree not greater than 4 over P. Two other problems (GR) and (GK) are studied: Is a ring R a finite algebra over a field P if (1) P belongs to the center of R and their units coincide, (2) R has a finite number of generators, (3) each "word" spelled by the generators is algebraic over P, and (4) (for (GK) only) each such word α has an inverse linearly expressible over P in powers of α? When the degrees of all elements are not greater than 3, the answer is affirmative, but a counter example shows that the answer to (GR) may be negative even when the degrees of all elements are bounded. *J. S. Frame* (Providence, R. I.).

Taussky, Olga and Todd, J. Inversion in groups. Quart. J. Math., Oxford Ser. **12**, 65–67 (1941). [MF 5217]

The transformation mapping the group-element x upon $tx^{-1}t$ is called the inversion with respect to the group-element t. B. de Kérékjárto [C. R. Acad. Sci. Paris **210**, 288–289 (1940); these Rev. **1**, 162] has obtained a characterization of the inversion with respect to the unit-element of an Abelian group without elements of order 2; and this criterion is generalized for any class of groups. Menger [Math. Z. **33**, 396–418 (1931)] has called the pair (ab^{-1}, ba^{-1}) the distance of the group-elements a and b; and the authors prove that the distance is invariant under inversions if, and only if, the group is either Abelian or a Hamiltonian 2-group. *R. Baer* (Urbana, Ill.).

Beaumont, Ross A. Projections of non-abelian groups upon abelian groups containing elements of infinite order. Amer. J. Math. **64**, 115–136 (1942). [MF 5993]

If there exists a biunivoque and monotonic increasing mapping of the set of all the subgroups of the group G upon the set of all the subgroups of the group H, then H is said to be a projection of G. If an abelian group contains elements of infinite order and possesses a non-abelian projection, then any two elements of infinite order are dependent (modulo the elements of finite order); and the author surveys those non-abelian groups that are projections of a direct product of an infinite cyclic group by an abelian group without elements of infinite order. It is shown that a non-abelian group H without elements of order 2 is a projection of an abelian group of the type described if, and only if, it satisfies the following conditions: (1) The elements of finite order in H form an abelian group F without elements of infinite order. (2) H/F is an infinite cyclic group. (3) If F contains two independent elements of order n, then all the elements of order n belong to the center of H. (4) If the element v in H generates H modulo F, and if H contains elements of order the prime number p, then there exists an integral p-adic number $i(p)$ such that $v^{-1}xv=x^{1+pi(p)}$ for every element x of order a power of p in H. If the group H may contain elements of order 2, then the conditions to be satisfied by H are somewhat more involved. *R. Baer* (Urbana, Ill.).

Szadowsky, L. E. Über die Strukturenisomorphismen von Freigruppen. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 171–174 (1941). [MF 5856]

It is shown that a biunivoque and monotonically increasing map f of the set of all the subgroups of the group G upon the set of all the subgroups of the group G' is induced by an isomorphism of G upon G' if there exists to every subgroup S of G which is generated by a finite number of generators an isomorphism of S which induces f on the subgroups of S. Furthermore, the outlines of a proof of the following theorem are indicated: if G is a non-commutative group such that all subgroups generated by a finite number of elements are free groups, if f is a biunivoque and monotonically increasing map of the set of all the subgroups of G upon the set of all the subgroups of a group G', then there exists one and only one isomorphism of the group G which induces f. *R. Baer* (Urbana, Ill.).

Liapin, Eugen. On the decomposition of Abelian groups into direct sums of rational groups. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 205–237 (1940). (Russian. English summary) [MF 3460]

This is an intensive and largely successful search for those properties of an Abelian group G which are equivalent to the existence of a decomposition of G into a direct sum of rational groups. A rational group (locally cyclic group, group of rank one) is isomorphic to a subgroup of the rationals, or of the rationals mod 1. An integer n is called a divisor of a coset A, subset of a group G, if there exists an $x \in G$ and $y \in A$ such that $y=nx$. A set of integers is called a characteristic if it consists of all integers of the form $\Pi p_i^{m_i}$, where p_i is the "ith prime" and $m_i \leqq \alpha_i$, the ith coordinate of the characteristic. Some of these α_i may be infinite. The set of divisors of a coset A constitute a characteristic, denoted by $\chi(A)$. Two characteristics α and β belong to the same equivalence class σ, called an overtype, if there exists an integer N such that $\alpha_i=\beta_i$ whenever $i \geqq N$ or $i \leqq N$ and one of these coordinates is infinite. An α and a β may be multiplied by the rule $\alpha \cdot \beta = \gamma$, where $\gamma_i=\alpha_i+\beta_i$, and this multiplication extends to overtypes.

A subgroup H of G is called non-extensible if $x \in G$, $nx \in H$ imply $x \in H$. The extension $R(K)$ of a set K of elements of G is the smallest non-extensible subgroup which contains all of K. With every abstract overtype σ and every non-extensible subgroup H of G there are associated the groups: (1) $H(\sigma)$, the extension of the set of elements of H whose overtypes are not a proper divisor of σ, (2) $\bar{H}(\sigma)$, the extension of the set of elements of H whose elements are not a divisor of σ, (3) $\tilde{H}(\sigma)$, the set of all elements of H whose overtypes are divisible by σ. With every coset A there is associated an overtype $\sigma=\sigma(A)$, determined, of course, by $\chi(A)$.

A coset A is called normal if it contains an element y such that $\chi(y)=\chi(A)$. This concept is elaborated in the following theorem: suppose that H is a non-extensible subgroup of G and A is a coset associated with H. In order that the coset A be normal it is necessary and sufficient that there exist a $y \in A$ such that simultaneously: $\sigma(y)=\sigma(A)$ and $\chi(A)=\chi(y+\tilde{H}(\sigma(y)))$. It is proved that if a non-extensible subgroup H of G and the factor group G/H are direct sums of rational groups, then the necessary and sufficient condition that H be a direct factor of G is that all cosets $x+H$, $x \in G$, be normal. Since the torsion subgroup (consisting of periodic elements) of a group G is non-extensible, the decomposition problem for a general group is reduced to that for torsion groups (solved by Prüfer, in the countable case) and for groups without periodic elements. This latter problem is solved in the countable case, as follows.

An element $x \in G$ is called regular if $\chi(x)=\chi(x+\bar{G}(\sigma(x)))$. The group G is said to admit a regular representation if every element of G can be expressed as a sum of regular elements corresponding to different overtypes. Now, let G be a countable group, σ an overtype. Let $g_{\sigma,1}, g_{\sigma,2}, \cdots$, represent the elements whose overtype is σ, in some enumeration. Define these groups: $G_{\sigma,1}=\bar{G}(\sigma)$, $G_{\sigma,n}=R(G_{\sigma,n-1}, g_{\sigma,n-1})$, $n=2, 3, \cdots$. Theorem: The countable group G with no elements of finite order is a direct sum of rational groups if and only if G admits a regular representation and every coset $g_{\sigma,n}+G_{\sigma,n}$ is normal (for all σ and n). The author shows, by a study of various examples, that these conditions, despite the element of arbitrary construction which occurs in them, are quite manageable. *L. Zippin.*

Thrall, R. M. Young's semi-normal representation of the symmetric group. Duke Math. J. 8, 611–624 (1941). [MF 5946]

A new derivation of A. Young's seminormal representation of the symmetric group is given. The method is based upon the construction of a complete set of primitive orthogonal idempotents of the corresponding group ring. Each of these idempotents corresponds to a regular Young diagram; they are defined by means of a simple recursive formula. The method can also be used in order to obtain the representations of the alternating group in unitary form. *R. Brauer* (Princeton, N. J.).

Nakayama, T. On some modular properties of irreducible representations of a symmetric group. I. Jap. J. Math. 18, 89–108 (1941). [MF 5199]

Let S_n denote the symmetric group of degree n. There exists a (1-1) correspondence between the irreducible representations F of S_n and Young's symmetry diagrams T consisting of n letters. The author determines the exact power of a prime p which divides the degree of a given representation F. Though we have an explicit formula for this degree, great combinatorial difficulties have to be overcome in finding the exponent of p. If T is the symmetry diagram, then the "hook" (i, k) is the set of all places (r, s) in T where either $r=i$, $s \geqq k$ or $r \geqq i$, $s=k$. The length of the hook is the number of places appearing here. If a hook H of length g is removed from T, and the letters below and to the right of H are pushed up and to the left, a new symmetry diagram $T-H$ consisting of $n-g$ letters is obtained. A p-hook is a hook whose length is a power $p^e \geqq 1$ of p. Let H_1 of length p^{a_1} be one of the longest p-hooks which appear in the diagram T belonging to the representation F. Then take one of the longest p-hooks H_2 in $T-H_1$, whose length p^{a_2} does not exceed p^{a_1}. After that take one of the longest p-hooks H_3 in $T-H_1-H_2$, whose length p^{a_3} does not exceed p^{a_2}, etc. If $e(m)$ denotes the exponent to which p divides the integer m, then the exact power of p in the degree f of F is given by $e(f)=e(n!)-\sum_i e(p^{a_i}!)$. In an appendix, Nakayama uses the hooks to give a neat formulation of Murnaghan's recursion formula for the characters of the symmetric group [cf. F. D. Murnaghan, Amer. J. Math. 59, 437–488 (1937)]. *R. Brauer* (Princeton, N. J.).

Nakayama, Tadasi. On some modular properties of irreducible representations of symmetric groups. II. Jap. J. Math. **17**, 411–423 (1941). [MF 5641]

In this paper, the author studies the modular characters of the symmetric permutation group S_n in n letters for a prime p, for which $n/2<p\leqq n$. The order $n!$ of S_n then is divisible only by the first power of p. For groups G of an order satisfying this condition, the reviewer has given rather complete results concerning the behavior of the ordinary and modular characters (mod p) [Proc. Nat. Acad. Sci. U. S. A. **25**, 252–258 (1939); cf. also the following review]. For $G=S_n$, the task remains to determine which place the characters, as described in this connection, have in the Frobenius-Young theory of the characters of the symmetric group. This problem is solved here; for example, the tables are determined for which the corresponding representations belong to the same p-block. In the case $p\leqq n/2$, the author makes a conjecture concerning the p-block distribution of the characters of S_n. *R. Brauer* (Princeton, N. J.).

Brauer, Richard. Investigations on group characters. Ann. of Math. (2) **42**, 936–958 (1941). [MF 5524]

If the order of a group $\mathfrak{G}$ is p^ag', where p is a prime and $(p, g')=1$, it has been proved by Brauer and Nesbitt [Ann. of Math. (2) **42**, 556–590 (1941); cf. these Rev. **2**, 309] that a representation of $\mathfrak{G}$ whose degree is divisible by p^a is absolutely irreducible as a modular representation of $\mathfrak{G}$. In this paper representations of $\mathfrak{G}$ whose degrees are divisible by p^{a-1} but not by p^a are studied. Amongst other results, it is shown that the degrees of all representations belonging to the same "block" B [loc. cit.] as one such representation are also divisible by p^{a-1}; B is said to be of type $a-1$. From this it follows without difficulty that the only simple group of order $4p^aq^b$ ($a\leqq 2$) is that of order 60; also, that the only simple groups of order $3p^aq^b$ ($a\leqq 2$) are those of orders 60 and 168. Two irreducible representations Z_λ, Z_μ of $\mathfrak{G}$ are said to be "p-conjugate" if the character ζ_λ of Z_λ can be carried into ζ_μ by a change in the p^a roots of unity which leaves the g'th roots unaltered. If z_λ is the degree of Z_λ, and if the number of its p-conjugate representations is r_λ, then it is shown that $r_\lambda z_\lambda\not\equiv 0 \pmod{p^{a+1}}$. If $r_\lambda z_\lambda\equiv 0 \pmod{p^a}$ then $z_\lambda\equiv 0 \pmod{p^a}$. Again, if $z_\lambda\equiv 0 \pmod{p^{a-1}}$ but $z_\lambda\not\equiv 0 \pmod{p^a}$ then r_λ divides $p-1$, and

$$r_1r_2\cdots r_w\left(\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_w}\right)=p,$$

where w is the number of families of p-conjugate representations appearing in the block B of type $a-1$. These results hold for p odd or even, though the proofs are somewhat different in this latter case. *G. de B. Robinson.*

Brauer, Richard. On the connection between the ordinary and the modular characters of groups of finite order. Ann. of Math. (2) **42**, 926–935 (1941). [MF 5523]

This paper continues the development of the theory of the modular characters of a finite group $\mathfrak{G}$ as begun by Brauer and Nesbitt [University of Toronto Studies, Math. Series, no. 4, 1937; Ann. of Math. (2) **42**, 556–590 (1941); cf. these Rev. **2**, 309]. The author introduces here generalized decomposition numbers $d_{\mu\nu}^i$ of $\mathfrak{G}$, such that

$$\zeta_\mu(P_iV_i)=\sum_{\nu=1}^{k_i} d_{\mu\nu}{}^i\varphi_\nu{}^i(V_i),$$

where ζ_μ is an ordinary character, P_i is an element of $\mathfrak{G}$ whose order is p^{a_i} and $\varphi_\nu{}^i$ is one of k_i distinct absolutely irreducible modular characters of the normalizer $\mathfrak{N}(P_i)$ of P_i, while V_i is a p-regular element of $\mathfrak{N}(P_i)$. The numbers $d_{\mu\nu}^i$ are integers in the field of the p^{a_i}th roots of unity. If P_i is replaced by $G^{-1}P_iG$, then the $d_{\mu\nu}^i$ are merely rearranged, so that they are invariants of $\mathfrak{G}$. The case $i=0$ is that previously studied, in which the decomposition numbers are ordinary integers but where the decomposition is restricted to p-regular elements of $\mathfrak{G}$, that is, to those whose orders are prime to p. *G. de B. Robinson.*

Kondô, Kôiti. Decomposition of the characters of some groups. I. Proc. Phys.-Math. Soc. Japan (3) **23**, 265–271 (1941). [MF 4766]

The characters of the n-dimensional hyper-octahedral group G_n have been determined by A. Young [Proc. London Math. Soc. **31**, 273–288 (1930)]. The author develops formulas analogous to the formulas of J. Schur and Nakayama for the recurrent calculation of the characters of the symmetric group [cf. T. Nakayama, Jap. J. Math. **18**, 89–108 (1941)]. The group G_n has two subgroups A_n and B_n of index 2 which are studied. The decompositions of the characters of A_n into those of A_{n-1} and of the characters of B_n into those of B_{n-1} are described. *R. Brauer.*

Lewis, Paul E. Characters of abelian groups. Amer. J. Math. **64**, 81–105 (1942). [MF 5991]

Let A and V be two Abelian groups, additively written. A character of A with values in V is a homomorphic mapping of A into V. The characters then again form an Abelian group $C_V(A)$. The paper is concerned with the question of under what conditions on A and V will one of the following relations hold true: (a) $C_V(A)\cong A$; (b) $C_V(S)\cong S$ for every subgroup S of A; (c) $C_V(A/S)\cong A/S$ for every subgroup S of A. In the first three sections the structure of Abelian groups with a (noncommutative) ring of operators R is studied; it is assumed that R has no divisors of 0 and that every ideal of R is a two-sided principal ideal. In particular, the properties of a quasibasis of the group are investigated. For the study of the characters, it is assumed that A and V admit the elements of R as left-operators, that V admits the elements of the anti-isomorphic ring R' as right-operators (with $s(vt')=(sv)t'$ for s in R, v in V, t' in R') and that the further assumptions are satisfied: (1) If v is any element of V, and (r) is a left ideal in R, then there exists a right ideal (s') in R' such that $(r)v=v(s')$ and $(s)\subseteqq r$, where s is the element of R corresponding to s'. (2) If v is an element of V and (r') is a right ideal in R, there exists a left ideal (s) in R satisfying $v(r')=(s)v$ and $(s')\subseteqq(r')$. Under this set up, a character of A in V is defined as a mapping $x\to f(x)$ of A into V such that $f(x+y)=f(x)+f(y)$, $f(rx)=f(x)r'$ for x, y in A, r in R. The order of an element x is defined as the left ideal consisting of all elements w of R such that $wx=0$; a group is cyclic if it is obtained from a fixed element by left multiplication with the elements of R. As an example of the results obtained, the following theorem may be quoted: Let A be a group without elements of order 0. Then $C_V(A)\cong A$ if and only if the following two conditions hold: (a) A is a direct sum of a finite number of cyclic groups; (b) if A contains an element of order (r), then the set of elements x of V with $rx=0$ forms a cyclic group of order (r). *R. Brauer* (Princeton, N. J.).

Topel, Bernard J. On isometric imbedding in Abelian groups. Rep. Math. Colloquium (2) 3, 47–55 (1941). [MF 5781]

Given a set of points S with "distances" between pairs of points defined as pairs of inverse elements of a group G, the author obtains the following conditions guaranteeing the existence of a G-isometric embedding of S (that is, a one to one correspondence between the elements of S and some of the elements of G such that distances in S are differences in G of the images of the pairs in S): (a) S contains two elements p and q whose distance does not involve an element of G of order two and each set of four points of S containing p and q can be G-isometrically embedded; or (b) S contains at least five points, no distances in S involve elements of G of order 2, 3 or 6, and each set of three points of S can be G-isometrically embedded. These conditions differ slightly from earlier results summarized in the paper.

C. B. Tompkins (Madison, Wis.).

Morita, Kiiti. A remark on the theory of general fuchsian groups. Proc. Imp. Acad. Tokyo 17, 233–237 (1941). [MF 5695]

The author gives an interesting generalization of Fuchsian groups by considering the set A_{nm} of all matrices of type (n, m) which satisfy the condition $E^{(m)}-\bar{Z}'Z>0$ (where $E^{(m)}$ denotes the unit matrix of order m). Consider the matrices U of order $n+m$ satisfying the equation

$$\bar{U}'S_{nm}U=S_{nm} \quad \text{with} \quad S_{nm}=\begin{pmatrix} E^{(n)} & 0 \\ 0 & -E^{(m)} \end{pmatrix};$$

let us write

$$U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

the types of U_1, U_2, U_3, U_4 being (n, n), (n, m), (m, n), (m, m), respectively. Consider then the transformation

$$W=(U_1Z+U_2)(U_3Z+U_4)^{-1}.$$

The important remark of the author is that the group Γ_{nm} of these transformations, or better the group $B_{nm}=\Gamma_{nm}/\Gamma_{nm}^{*}$, can be considered as the group of displacements in the space A if Γ_{nm}^{*} is the group of matrices inducing the identical displacement. The subgroups of B_{nm} without infinitesimal transformations are Morita's Fuchsian groups. The author proves that they are properly discontinuous and that the corresponding θ-series are convergent. He also studies the "distances" in the space A_{nm}.

G. Fubini.

Whitehead, George W. Homotopy properties of the real orthogonal groups. Ann. of Math. (2) 43, 132–146 (1942). [MF 6066]

Let R_n be the group of rotations of the n-sphere S^n. In this paper the homotopy groups $\pi_i(R_n)$ are determined for $i\leqq 5$ and all n. The known facts about $\pi_i(S^n)$ are used; the methods follow the lines of a paper by Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U. S. A. 27, 60–64 (1941); these Rev. 2, 323]. The relative homotopies of Hurewicz are employed, in particular, of R_n relative to R_{n-1}. Let I, I_2, $+$ denote integers, integers mod 2 and direct sum, respectively. It is shown that $\pi_3(R_n)\approx I$ ($n=2$ and $n\geqq 4$), $\pi_3(R_3)\approx I+I$, $\pi_4(R_3)\approx\pi_4(R_4)\approx I_2$, $\pi_4(R_3)\approx I_2+I_2$, $\pi_5(R_3)\approx I$, $\pi_4(R_n)\approx\pi_5(R_n)\approx 0$ for other n. The generators of the groups are given explicitly. One important application is that, if $n\equiv 1 \pmod 4$, then any two vector fields over S^n are somewhere dependent.

H. Whitney (Cambridge, Mass.).

Weyl, Hermann. On the use of indeterminates in the theory of the orthogonal and symplectic groups. Amer. J. Math. 63, 777–784 (1941). [MF 5625]

Let $(x_1, \cdots, x_\nu, x_1', \cdots, x_\nu')$ denote a general vector in 2ν dimensions and $[xy]$ the skew symmetric bilinear form $\sum(x_\alpha y_\alpha'-x_\alpha' y_\alpha)$. The symplectic group is the group of linear transformations that leave $[xy]$ invariant. A general infinitesimal symplectic transformation has the form $dx=Sx$, where S is a matrix with $n(n+1)/2$ indeterminate elements such that $I^{-1}S^{*}I=-S$, where * denotes the transposed and I is the matrix of $[xy]$. The author presents here some improvements and a correction in the treatment of the vector invariants and of the representations of the symplectic group that appears in his book "The Classical Groups" [Princeton University Press, Princeton, N. J., 1939; these Rev. 1, 42]. The fundamental theorem for infinitesimal vector invariants is now proved directly using a new "Capelli" identity. If $A=(E-S)(E+S)^{-1}$, where E is the identity matrix, then the direct product $\Pi_f(A)=R(s)\,|E+S|^{f}$, where $R(s)=\sum C_\rho\sigma_\rho$, σ_ρ monomials in the parameters s_{ij} of S. It is shown that the set of matrices $\sum\lambda_\rho C_\rho$, λ_ρ in a field k, is an algebra C_f. This algebra is completely reducible if k is the field of rational numbers. The same remarks hold for $\Pi^{(f)}(A)$, the representation that decomposes into Π_0, Π_1, $\cdots$, Π_f. It follows that $C^{(f)}$ defined by $\Pi^{(f)}(A)$ is the enveloping algebra of the matrices $\Pi^{(f)}(B)$, where B ranges over the symplectic group over any field of characteristic 0. These results hold also for the orthogonal group. The correction of the book concerns the statement of Theorem 5.3 B and the corollary to 7.1 B.

N. Jacobson.

Ore, Oystein. Theory of monomial groups. Trans. Amer. Math. Soc. 51, 15–64 (1942). [MF 6088]

Let H be a given group. A monomial transformation of m indeterminates $x_1, x_2, \cdots, x_m$ over H is a transformation $x_\mu\to r_\mu x_{i_\mu}$, where $(i_1, i_2, \cdots, i_m)$ forms a permutation of $(1, 2, \cdots, m)$ and where r_μ is an element of H; the multiplication of r_μ and x_{i_μ} is to be taken in a purely formal manner. The multiplication of monomial transformations then can be defined in a natural manner. All monomial transformations form a group $\Sigma_m(H)=\Sigma_m$, the complete monomial group or "symmetry" of degree m over H. The subject of the paper is an exhaustive study of this group $\Sigma_m(H)$. In the first chapter the simplest properties are discussed, for example, the distribution of the elements into classes of conjugate elements is given, and the centralizer of a given element is determined. The monomial permutations of the form $x_\mu\to r_\mu x_\mu$ (r_μ in H) form a normal subgroup V_m, the basis group; for $m\geqq 3$, the subgroup V_m is characteristic. On the other hand, the transformations of the type $x_\mu\to x_{i_\mu}$ form a subgroup S_m which is isomorphic to the symmetric group on m letters. We have $\Sigma_m=V_mS_m$ and $V_m\cap S_m=E$, so that Σ_m splits over V_m. All subgroups T of Σ_m are determined, for which $\Sigma_m=V_mT$, $V_m\cap T=E$. They are all conjugate to S_m if and only if H does not contain a subgroup different from E which is homomorphic to the symmetric group on $m-1$ letters.

The second chapter deals with the construction of all normal subgroups of Σ_m. In the general case, a large number of normal subgroups occurs. The commutator group Σ_m' of Σ_m is determined; for $m\geqq 5$, the second commutator group Σ_m'' is equal to Σ_m'. In the third chapter, the author succeeds in the construction of all automorphisms of Σ_m with the exception only of the case $m=6$. The exceptional rôle of $m=6$ is explained by the exceptional behavior of the

automorphisms of the symmetric group S_4. In the final chapter, the monomial representations of an arbitrary group G are investigated. All transitive monomial representations can be constructed [cf. W. Turkin, Math. Ann. 111, 743–747 (1935)]. The group H will be a factor group H_0/K, where H_0 is a subgroup of G and where K is a normal subgroup of H_0. Thus G is mapped homomorphically into $\Sigma_m(H_0/K)$. The normalizer and the centralizer of the image in $\Sigma_m(H_0/K)$ are determined. *R. Brauer.*

Freudenthal, Hans. Die Topologie der Lieschen Gruppen als algebraisches Phänomen. I. Ann. of Math. (2) **42**, 1051–1074 (1941). [MF 5816]

A real simple non-Abelian group is said to be of the first kind if its complex extension is not simple; it is of the second kind if it remains simple upon extension. The author proves the important theorem that every isomorphism between simple groups of the second kind is necessarily continuous. The prerequisite for the proof is a strictly group-theoretical description of the groups of the first kind and the 1-dimensional compact germs of subgroups of a group of the second kind [Theorems 4′ and 6]. A strictly group-theoretical determination of a continuous group means, roughly speaking, the description of the elements in the group by products of commutators and that of groups by algebraic concepts. Thus the formal constructions imply topological facts. The algebra of Lie rings combined with the exponential representation of compact neighborhoods of the identity furnish the key for the solution of the author's problem [Theorems 5, 6 and 7]. For the explicit technical details see the original paper. *O. F. G. Schilling.*

***Weil, André. L'intégration dans les groupes topologiques et ses applications.** Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940. 158 pp.

[This book has been republished by the author at Princeton, N. J., 1941. $1.75.]

The author discusses those problems in the theory of topological groups which center around the notion of Haar measure. It is a clear exposition of the present state of the theory, and contains most of the important results which were recently obtained in this field. These results are completed in many points, and the methods of proof are greatly improved. On almost all pages the reader will find new ideas of the author. The book consists of seven chapters and two appendices. At the end of each section there are historical remarks, which, together with the complete references at the end of the book, give a clear and comprehensive view of the recent development of the theory.

In Chapter I the general theory of topological groups is given. The exposition follows the style of N. Bourbaki [Topologie générale, Actual. Sci. Ind., no. 858, Paris, 1940; cf. these Rev. 3, 55]. The topological space of a topological group is treated as a uniform space in the sense of the author [Sur les espaces à structure uniforme et sur la topologie générale, Actual. Sci. Ind., no. 551, Paris, 1937]. Separability is not assumed here nor elsewhere in the book. Special attention is paid to compact (=bicompact) groups and compact subsets of a group. An elegant exposition of factor groups, direct product groups and of homogeneous spaces (that is, co-set spaces obtained from a topological group by closed subgroups) is given. Projective limit groups (that is, topological groups which are defined as a Moore-Smith limit of a directed system of topological groups with a projective system of homomorphic mappings) are discussed by means of the ideas of C. Chevalley, using the method of filters due to H. Cartan. These projective limit groups generalize the sequential limit groups of H. Freudenthal and L. Pontrjagin.

In Chapter II the existence and uniqueness of (left invariant) Haar measure is proved on a locally compact group G. The Banach spaces $L^p(G)$ $(1\leqq p\leqq\infty)$ are introduced, and their properties investigated. ($L^\infty(G)$ is the completion of the space of all real valued continuous functions $f(x)$ defined on G which vanish outside some compact subset of G, with $\|f\|=\sup_{x\in G}|f(x)|$ as its norm.) It is proved that, for any $f(x)\in L^p(G)$ $(1\leqq p\leqq\infty)$, the mapping $a\to f_a(x)=f(a^{-1}x)$ is continuous on G to $L^p(G)$. $U_a(f)=f_a$ defines a unitary transformation U_a of $L^2(G)$, and $\{U_a\}$ $(a\in G)$ is a group topologically isomorphic to G with respect to the strong topology in the space of bounded linear transformations of $L^2(G)$. Haar measure is treated as a left invariant integral $I(f)=\int_G f(x)dx$ (that is, $I(f_a)=I(f)$ for all $a\in G$) defined on $L^\infty(G)$. This treatment is technically better than that of A. Haar, and makes it possible to apply the method of weak topology. In the existence proof, Tychonoff's theorem is used instead of the axiom of choice or Banach limit. The proof of the uniqueness is entirely different from that of J. von Neumann [Rec. Math. [Mat. Sbornik] N.S. 1 (43), 721–734 (1936)]. (Following the ideas of the author, H. Cartan [cf. the following review] recently gave a proof by which the existence and the uniqueness are proved at the same time without using the axiom of choice.) Relations between left and right invariant Haar measures are discussed. Unimodular groups (that is, groups on which left and right invariant Haar measures coincide, for example, compact groups or semi-simple Lie groups) are discussed in detail. Finally, the existence and uniqueness of an invariant measure is proved on a homogeneous space which is obtained from a locally compact group.

In Chapter III the author first shows that the left invariant Haar measure of $G\times G$ is invariant under the transformations: $(x, y)\to(yx, y)$, $(x, y)\to(y^{-1}x, y)$, etc. Then the convolution $h=f^*g$ of $f(x)$, $g(x)\in L^1(G)$ is defined by: $h(x)=\int_G f(y)g(y^{-1}x)dy$; $h(x)$ belongs to $L^1(G)$ and is uniformly continuous in x; $L^1(G)$ becomes a normed ring with respect to this operation, and is called the group ring of G. For any measurable subsets A, B of G of finite measure, $m(A\cap xB)$ is a uniformly continuous function of x, belonging to $L^1(G)$ and satisfying the following important formula: $\int_G m(A\cap xB)dx=m(A)\cdot m(B^{-1})$. From this follows that if A is measurable and of positive measure, then so is A^{-1}, and further that $A^{-1}A$ and AA^{-1} contain an open neighborhood of the unit element of G. This fact makes it possible to introduce a topology on G starting from an invariant measure of G. Finally, positive definite functions are introduced on a locally compact group. Bochner's theorem on the integral representation of positive definite functions is extended, in Chapter VI, to the case of functions on a locally compact Abelian group. These and related problems have been investigated independently by I. Gelfand, M. Krein and D. Raikov.

Chapter IV discusses the general theory of representations. Schur's lemma is proved, and the reducibility of the representations of direct product group is investigated. It is proved that every measurable representation of a locally compact group is continuous, and that every bounded representation is equivalent to a unitary representation.

Chapter V is devoted to the analysis of compact groups. Peter-Weyl's theorem plays a fundamental rôle here. Fol-

lowing the ideas of E. Cartan and H. Weyl, the results are extended to the case of compact homogeneous spaces. The Frobeniusean theory of induced representations concerning finite groups is generalized to the case of compact groups and their subgroups. Among other things it is shown that every irreducible representation of a closed subgroup g of a compact group G can be obtained by reducing a representation of g which is induced by an irreducible representation of G. Then the center of the group ring of a compact group is investigated. Finally, extending the results of J. von Neumann and L. Pontrjagin, the structure of (not necessarily separable) compact groups is determined. Every compact group is a projective limit of compact Lie groups. Every compact connected group of finite dimension is of the form: $(L\times R^n\times D)/\nabla$, where L is a semi-simple compact Lie group, R^n is a finite dimensional vector group, D is a projective limit group of a sequence of finite Abelian groups (and hence totally disconnected) and ∇ is a discrete subgroup of the center of $L\times R^n\times D$ with a finite number of generators. From this follows that every compact connected and locally connected group of finite dimension is a Lie group.

The duality theory and the structure of locally compact Abelian groups are investigated in Chapter VI. The results of L. Pontrjagin and E. R. van Kampen are obtained by a simple new method. Among other things it is shown that every locally compact Abelian group G has a discrete subgroup g such that the factor group G/g is compact. The Fourier transform is defined for functions of $L^2(G)$ on a general locally compact Abelian group G, and Plancherel's theorem is extended to the general case. This theorem is proved in the general case by showing that, if it is true for a closed subgroup g and the factor group G/g of a locally compact Abelian group G, then it is also true for G itself. If G is the real line, then this method gives a new proof of the classical Plancherel theorem. We have only to take g as the group of all integers.

In Chapter VII almost periodic functions are discussed. A result of the author [C. R. Acad. Sci. Paris **200**, 38–40 (1935)] is generalized to show that, for every maximally almost periodic group G, there exists a compact group G^* which contains G as a dense subgroup and such that every continuous real valued almost periodic function defined on G can be uniquely extended to a continuous (and hence almost periodic) function on G^*. This shows that the theory of almost periodic functions is equivalent to the theory of continuous homomorphisms of a given group into a compact group. Thus the problem of the existence and uniqueness of the mean of an almost periodic function is reduced to the same problem concerning Haar measure on a compact group. In order that a connected locally compact group G be maximally almost periodic, it is necessary and sufficient that G be a direct product of a compact group and a finite dimensional vector group. This extends a result of H. Freudenthal concerning the case when G is separable. Finally, almost periodic functions on Abelian groups, and in particular, those on the real line, are investigated in detail.

In the first appendix, the following result is obtained: Let G be a group without topology, and let $m(E)$ be a left invariant measure defined on G such that $f(y^{-1}x)$ is measurable on $G\times G$ whenever $f(x)$ is measurable on G. If there exists, for any measurable subset A of G of positive finite measure, an $x\varepsilon G$ such that $m(A\cap xA)<m(A)$, then it is possible to introduce a topology into G in such a way that the completion G^* of G with respect to this topology is locally compact, and that the left invariant Haar measure on G^* induces the given measure $m(E)$ on G. Such a topology is unique, and can be obtained by taking the neighborhood system of the unit element as the family of all sets of the form: $A^{-1}A$ (or AA^{-1}), where A is an arbitrary measurable subset of G of positive measure. The same topology can be obtained by considering a group $\{U_a\}$ $(a\varepsilon G)$ of unitary transformations on $L^2(G)$ (defined in terms of the measure $m(E)$) which is algebraically isomorphic to G, and by taking the strong topology of the space of bounded linear transformations of $L^2(G)$. This result was announced earlier by the author [C. R. Acad. Sci. Paris **202**, 1147–1149 (1936)], and meanwhile the same problem was discussed by K. Kodaira [Proc. Phys.-Math. Soc. Japan **23**, 67–119 (1941); these Rev. **2**, 317]. In the second appendix, another proof of the uniqueness of Haar measure is given. The method of proof is similar to that given independently by J. von Neumann [Princeton lecture, 1940–1941].

S. Kakutani (Princeton, N. J.).

Cartan, Henri. Sur la mesure de Haar. C. R. Acad. Sci. Paris **211**, 759–762 (1940). [MF 5368]

A new proof is given of the existence and the uniqueness of a left invariant Haar measure on a locally (bi)compact topological group G. The main point is that both of these are proved at the same time without the axiom of choice. The method of proof is analogous to the one given by A. Weil [cf. the preceding review]. A fundamental rôle is played by the following lemma: Let $f(x)$ be a non-negative continuous function defined on G which vanishes outside a compact set, and let V be a neighborhood of the unit element such that $y^{-1}x\varepsilon V$ implies $|f(x)-f(y)|<\epsilon$. Then, for any $\alpha>\epsilon$ and for any non-negative continuous function $g(x)$ which vanishes outside V, there exists a linear combination $h(x)=\sum_{i=1}^{n}c_i g(s_i^{-1}x)$, $c_i>0$, $s_i\varepsilon G$, $i=1,\cdots,n$, such that $|f(x)-h(x)|<\alpha$ for all $x\varepsilon G$. *S. Kakutani.*

Rees, D. Note on semi-groups. Proc. Cambridge Philos. Soc. **37**, 434–435 (1941). [MF 5548]

In a previous paper [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127] a semi-group was defined as simple if it contained no proper ideals. It was completely simple if it satisfied in addition the following two conditions: (i) for each x there exists idempotent elements e and f such that $ex=x$ and $xf=x$; (ii) there exists at least one idempotent element $e\neq 0$ (called primitive) such that $ex=e$ or $xe=e$ implies that $x=e$. The present note proves that in a simple semi-group the second condition implies the first. *H. H. Campaigne* (Minneapolis, Minn.).

Clifford, A. H. Semigroups admitting relative inverses. Ann. of Math. (2) **42**, 1037–1049 (1941). [MF 5530]

A semigroup S is said to admit relative inverses if for each element a there are elements e and a' in S such that $ea=a=ae$, and $a'a=e=aa'$. A semigroup admits relative inverses if and only if it is the class sum of mutually disjoint groups. Another characterization is that it is the class sum of mutually disjoint completely simple semigroups S_α without zeros, such that they form a semilattice under multiplication. A simple semigroup without zero admits relative inverses if and only if it is completely simple. A construction is given (in terms of a semilattice, a set of groups G_α and a set of homomorphisms $\phi_{\alpha\beta}$ from G_α on G_β) for all semigroups which admit relative inverses and in which every pair of idempotent elements commute. In this case the S_α

of the third sentence above are groups. Evidently the condition that the idempotent elements commute is a stringent one, for the construction does not carry over to the general case. *H. H. Campaigne* (Minneapolis, Minn.).

Borůvka, O. Über Ketten von Faktoroiden. Math. Ann. 118, 41–64 (1941). [MF 6309]

Consider a groupoid in the sense of Hausmann and Ore, that is, a set of elements with a single-valued multiplication among them. Partition the groupoid, getting another, which is called a factoroid. The only restriction on the partition (called generating) is that the product of two sets must be contained entirely within some other set of the partition. It is shown that these generating partitions form a complete lattice. The upper bound of all is that partition with just one set, and the lower bound is that in which every set has but one element. A groupoid is simple if its only generating partitions are those two just mentioned. It is relatively simple with respect to an element a if each of its generating partitions (except the upper bound) has the set with the single element a. A factoroid, being a groupoid, can be simple or relatively simple. The preservation of generating partitions under homomorphisms and isomorphisms is discussed.

If $\mathfrak{A}$ is a factoroid with B one of its elements, and if B is partitioned so as to form the factoroid $\mathfrak{B}$, then $\mathfrak{B}$ is said to be eingegliedert in $\mathfrak{A}$. If $\mathfrak{C}$ is eingegliedert in $\mathfrak{B}$, and $\mathfrak{D}$ in $\mathfrak{C}$, and so forth to $\mathfrak{Z}$, then we have a chain from $\mathfrak{A}$ to $\mathfrak{Z}$. The number of factoroids in the chain is its length. These concepts are analogous to those in group theory. A chain $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \cdots$ is relatively simple if each $\mathfrak{A}_\gamma$ is relatively simple with respect to $\mathfrak{A}_{\gamma+1}$. It is simple if each $\mathfrak{A}_\gamma$ is simple. The important, but complicated, idea of adjoint chains is introduced, one of their properties being that they have the same ends. All of these concepts are used to prove an analogue of the Jordan-Hölder-Schreier theorem: If $\mathfrak{A}$ and $\mathfrak{C}$ are adjoint chains, then there are refinements of $\mathfrak{A}$ and $\mathfrak{C}$, respectively, which are isomorphic. He concludes with the statement: If one of these chains is relatively simple, then its reduced length is not greater than that of the other.

H. H. Campaigne (Washington, D. C.).

ANALYSIS

Theory of Functions of Complex Variables

Friedrichs, K. O. Integration in the complex plane. Elec. Engrg. 61, 139–143 (1942).

This is the second of a group of five lectures given before the American Institute of Electrical Engineers as a symposium on "Advanced Mathematics as Applied to Electrical Engineering."

Menger, Karl. Redundancies in the classical treatment of the Cauchy-Riemann conditions. Rep. Math. Colloquium (2) 2, 45–48 (1940). [MF 5775]

The author shows that there is a redundancy in the classical proof of the fact that $\partial u/\partial y=\partial v/\partial x$ ensures that $\int udx+vdy$ is independent of the path. *W. T. Martin.*

Milgram, A. N. A generalization of the Cauchy-Riemann equation. Rep. Math. Colloquium (2) 3, 28–30 (1941). [MF 5778]

The author derives a new condition which he shows is equivalent to the customary ones for the exactness of $udx+vdy$. The condition involves the existence at each point of the partial derivatives of certain functions associated with u and v. *W. T. Martin* (Cambridge, Mass.).

Broggi, U. Sulle serie di potenze. Boll. Un. Mat. Ital. (2) 2, 455–460 (1940). [MF 5565]

The author states the theorem: In order that the power series $c_0+c_1z+c_2z^2+\cdots$ whose radius of convergence is equal to 1 "have no singular point on its circle of convergence except $z=1$" it is necessary and sufficient that for all sufficiently small positive ϵ

$$\limsup \left| c_n-\binom{n}{1}\frac{\epsilon}{1+\epsilon}c_{n-1}+\cdots+(-1)^{n-1}\left(\frac{\epsilon}{1+\epsilon}\right)^{n-1}c_1\right|^{1/n}=\frac{1}{1+\epsilon}.$$

This theorem is false. What the author's proof really shows is a different theorem obtained from the foregoing statement by substituting for the phrase in quotation marks the following: "be regular in some circle of radius greater than 1 that contains the unit circle and is tangent to it at the point $z=1$." The author fails to observe that these two phrases have different meanings. He uses in his proof a variant of the Euler transformation, familiar in this subject since the work of Pringsheim, and employs the same method for deriving some results on entire functions [which could be obtained also as very special cases of known results on the singularities of the series $\sum\phi(n)z^n$, where $\phi(x)$ is an entire function of exponential type]. *G. Pólya.*

Broggi, Ugo. Sulle serie di potenze a coefficienti positivi decrescenti. Boll. Un. Mat. Ital. (2) 3, 7–9 (1940). [MF 5569]

Applying the theorem stated in the foregoing review the author obtains a theorem that may be stated thus. "If the sequence of positive numbers $c_0, c_1, c_2, \cdots, c_n, \cdots$ is totally monotone, in the sense of Hausdorff, and $\limsup |c_n|^{1/n}=1$ (for $n\to\infty$), then $z=1$ is the only singular point on the circle of convergence of the power series $c_0+c_1z+c_2z^2+\cdots$." [More than this is known; the analytic continuation of the power series is regular in the whole plane, except possibly on that part of the real axis that contains the positive numbers not less than 1; see Messenger of Math. 55, 189–192 (1926).] *G. Pólya* (Northampton, Mass.).

Western, D. W. An extension of the Laurent expansion. Amer. Math. Monthly 48, 444–450 (1941). [MF 5537]

If $f(z)$ is analytic in a certain annular region S, then $2\pi i f(z)$ can be expanded in a series of the form

$$\sum_0^\infty \int_C K(z,t)f(t)dt+\sum_0^\infty \int_C K(t,z)f(t)dt,$$

where $K(z,t)=(z+ht)^n(t+ht)^{-n-1}$, h is any non-negative number and C is a contour in S about the origin. This result is really an approximation by polynomials rather than an expansion in the sense of a Laurent expansion, since the terms of the series are polynomials in z or $1/z$ all of whose coefficients depend on $f(z)$; but it reduces to the Laurent expansion if $h=0$. A geometric characterization of the region S is given. *P. W. Ketchum* (Urbana, Ill.).

Miranda, Carlo. Su talune serie di funzioni olomorfe. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 829–836 (1941). [MF 5756]

The problem is to expand any function analytic at the origin in series of the form $\sum a_n f_n(z)$, where the $f_n(z)$ are given functions with zeros of order n at the origin. The main theorem is identical with the second part of Theorem 4.1 of R. P. Boas' recent paper [Trans. Amer. Math. Soc. **48**, 467–487 (1940); these Rev. **2**, 80]. The proof is different from that of Boas. Some special cases are also studied.

P. W. Ketchum (Urbana, Ill.).

Walsh, J. L. Note on the coefficients of overconvergent power series. Bull. Amer. Math. Soc. **48**, 163–166 (1942). [MF 6197]

As an addition to Ostrowski's classical theorem on overconvergent power series the following result is obtained. Let the power series $\sum c_n z^n$ with $|z|=1$ as radius of convergence display the "gaps" (m_k, n_k) and let the sequence of the partial sums $s_{m_k}(z)$ be overconvergent. Then, if $R_0>1$ [$r_0<1$] is arbitrary, a number $\sigma=\sigma(R_0)$, $0<\sigma<1$ [$\tau=\tau(r_0)$, $0<\tau<1$] exists such that, for every sequence $\mu_R, \mu_R \leqq m_R$ [$\nu_R, \nu_R \geqq n_R$],

$$\limsup \mu_R^{-1} \log |c_{\mu_k}| \leqq \{\sigma \limsup (m_R/\mu_R)-1\} \log R_0$$
$$[\limsup \nu_R^{-1} \log |c_{\nu_k}| \leqq \{\tau \liminf (n_R/\nu_R)-1\} \log r_0].$$

G. Szegő (Stanford University, Calif.).

Rathnam, P. The derivatives of an algebraic function. J. Indian Math. Soc. (N.S.) **5**, 145–147 (1941). [MF 6134]

A formula is obtained here for the derivative $u'(z)$ of an integral algebraic function $u(z)$ defined by an irreducible polynomial equation $f(z, u)=0$ having roots in u of multiplicity at most $m=3$ for all points z of the extended plane. The formula, which expresses $u'(z)$ in terms of its principal part at the branch points of $u(z)$, is a generalization from $m=2$ to $m=3$ of a similar formula given by S. Beatty [Trans. Roy. Soc. Canada. Sect. III. **25**, 79–82 (1931)].

M. Marden (Milwaukee, Wis.).

Shah, S. M. A note on the classification of integral functions. Math. Student **9**, 63–67 (1941). [MF 6001]

There are five possible classes of entire functions of given positive integral order ρ, determined by well-known relations among the numbers ρ, ρ_1, p, q [notation as in Titchmarsh, Theory of Functions, Oxford, 1932, chap. 8]. The author considers for each class what possibilities there are for the type of a function belonging to the class, and gives suitable examples to show that the possibilities which he does not exclude can actually occur. *R. P. Boas, Jr.*

Biggeri, Carlos. On the Julia lines of entire functions. Bol. Mat. **14**, 264–265 (1941). (Spanish) [MF 5869]

The author states sufficient conditions, one of which involves the elliptic modular function, for a line to be a Julia line of an entire function. He states as special cases the following results. If $P(z)$ is an odd polynomial with real coefficients, then $e^{P(z)}$ has the imaginary axis as a Julia line. Let $P(z)$ and $Q(z)$ be polynomials, with $Q(z)$ of degree less than six times that of $P(z)$. Then if the real part of $P(z)$ is bounded on a line, the line is a Julia line for $Q(z)e^{P(z)}$; if the real part of $P(z)$ approaches zero on a line, the line is a Julia line for $Q(z)\exp[e^{P(z)}]$ and $Q(z)\exp\{\exp[e^{P(z)}]\}$.

R. P. Boas, Jr. (Chapel Hill, N. C.).

Chandrasekharan, K. On Hadamard's factorization theorem. J. Indian Math. Soc. (N.S.) **5**, 128–132 (1941). [MF 6132]

Hadamard's factorization theorem for an entire function $f(z)$ of finite order ρ is proved by considering the integral

$$\int \frac{f'(z)}{f(z)} \frac{1}{z^p(z-w)} dz$$

taken over a set of expanding circles; here $f(0)\neq 0$ and $\mu=[\rho]$. Calculation of residues leads to a formula for $f'(w)/f(w)$ which is the logarithmic derivative of Hadamard's formula. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Rajagopal, C. T. A proof of Hadamard's factorization theorem. Math. Student **9**, 68–72 (1941). [MF 6002]

The author proves Hadamard's well-known factorization theorem from considerations based on the structure of a canonical product. *N. Levinson* (Cambridge, Mass.).

Rajagopal, C. T. Carathéodory's inequality and allied results. Math. Student **9**, 73–77 (1941). [MF 6003]

The author proves Carathéodory's inequality using, as he states, Cauchy's integral formula but not the maximum modulus principle. However the alternative relations he must derive from the Cauchy formula are more involved than the derivation of the maximum modulus principle.

N. Levinson (Cambridge, Mass.).

Onofri, Luigi. Sulle funzioni univalenti in una corona circolare. Boll. Un. Mat. Ital. (2) **3**, 113–115 (1940). [MF 5581]

Continuing some investigations of L. Spacek [Časopis Pěst. Mat. Fys. **62**, no. 2, 12–19 (1932)], the author shows that, if $f(z)$ is holomorphic in the annular region $A: r<|z|<R$ $(r>0)$, and

$$0<\Re\{e^{i\alpha} zf'(z)/f(z)\}<2\,|\cos\alpha|$$

for some real α, then $f(z)$ is univalent in A. An inequality is obtained for the coefficients in the Laurent series for $f(z)$.

H. S. Wall (Evanston, Ill.).

Joh, Kenzo. Theorems on "schlicht" functions. V. On the coefficient problem. Proc. Phys.-Math. Soc. Japan (3) **23**, 409–423 (1941). [MF 5405]

[The last communication appeared in the same Proc. **22**, 329–343 (1940); cf. these Rev. **1**, 308.] Suppose that the function

$$(1) \qquad f(z)=z+c_2z^2+\cdots+c_nz^n+\cdots$$

is regular and schlicht for $|z|<1$, that $0<\alpha\leqq 1$, and that η is a complex number for which $|\eta|=1$. Let $\zeta(\alpha|z)$ be the function determined by the equation

$$\frac{\zeta}{(1-\zeta)^2}=\alpha\frac{z}{(1-z)^2},$$

write $\zeta(\alpha, \eta|z)=\eta\zeta(\alpha|\eta z)$, $\zeta_k(z)=\zeta(\alpha_k, \eta_k|z)$, and let

$$\phi_0(z)=\frac{z}{(1-z)^2};$$

$$\phi_k(z)=\frac{\phi_{k-1}(\zeta_k(z))}{\zeta_k'(0)}=z+a_2^{(k)}z^2+\cdots+a_n^{(k)}z^n+\cdots.$$

Suppose that n is fixed, and that $f(z)$ is a function (1) for which c_n is extremal. Given $\epsilon>0$, J. Basilewitsch [Rec. Math. [Mat. Sbornik] N.S. **1** (**43**), 211–228 (1936)] has

shown that $|c_n - a_n^{(k)}| < \epsilon$ for $k > k_0(\epsilon, f)$ and suitable $\bar{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\bar{\eta} = (\eta_1, \eta_2, \cdots, \eta_k)$, where, moreover, we may suppose that $\bar{\alpha} \subset N_1(k)$, $N_1(k)$ being a neighborhood (2) $1 - \lambda_j < \alpha_j \leqq 1$ $(j = 1, 2, \cdots, k)$, with $\lambda_j = o(1)$ as $k \to \infty$. The author proves, for $n = 3, 4$, that

$$(3)\quad \Re\left\{\frac{\partial}{\partial \alpha_j} a_n^{(k)}\right\}_{\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1} > 0, \qquad j = 1, 2, \cdots, k;\ k \geqq 1,$$

unless $\eta_j = 1$; and then from (2), (3), and the obvious identities

$$[a_n^{(k)}]_{\alpha_1 = \alpha_2 = \cdots = \alpha_k} = [a_n^{(k)}]_{\eta_1 = \eta_2 = \cdots = \eta_k = 1} = n,$$

concludes that

$$(4)\qquad \Re\{a_n^{(k)}\} \leqq n, \qquad n = 3, 4$$

(which is equivalent to $|c_n| \leqq n$ for $n = 3, 4$). But there is a gap in the reasoning. For, although there is a neighborhood $N_2(k)$: $1 - \delta_j < \alpha_j \leqq 1$, $\delta_j = \delta_j(k)$ $(j = 1, 2, \cdots, k)$, in which (3) is valid, it is not shown that $N_1(k) \subset N_2(k)$. [This gap does not occur in the author's proof, which is correct, that $|c_2| \leqq 2$.]

The reviewer (using different methods) found that

$$(5)\quad \left[\frac{\partial a_n^{(k)}}{\partial \alpha_j}\right]_{\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1} = n(n-1) + 2\sum_{\mu=1}^{n-1} (-1)^{n-\mu} \cdot \mu^2 \cdot \eta_j^{n-\mu}, \qquad j = 1, 2, \cdots, k;\ n \geqq 1,$$

from which (3), for any $n > 1$, may be easily deduced. The "gap" is therefore equivalent to the conjecture that $|c_n| \leqq n$.

D. C. Spencer (Cambridge, Mass.).

Kawakami, Yosiro. On an extension of Löwner's lemma. Jap. J. Math. 17, 569–572 (1941). [MF 5648]

The author proves the following theorem: A domain D with boundary C included in the unit circle $|z| < 1$ in the z-plane is represented on the unit circle $|w| < 1$ in the w-plane by a simple function $w = F(z)$ such that $F(0) = 0$. If a closed set B on C on $|z| = 1$ is represented on a closed set B' on $|w| = 1$, then we have meas $(B) \geqq$ meas (B'), the equality holding only when D coincides with the unit circle if B is of positive measure. This theorem extends one of Löwner, which proves the same result in the case C is a simple curve and B a segment thereof. For the proof the author makes use of a "hypofunction" $\underline{H}_D'$ defined as follows: Let f be a bounded function on C and F_i the class of subharmonic functions in D, none of whose limiting values on C exceed f. Then $\underline{H}_D'$ is the superior envelope of the functions of F_i. It is harmonic in D and is equal to $\int_C f(Q)\, d\mu(e_Q, P)$, where μ is the mass distribution obtained from sweeping out unit mass from P onto C. In case f is the characteristic function of a closed set B, this hypofunction is a generalization of harmonic measure. Using the integral representation for H and the invariance of the subharmonic property under conformal transformation, the author obtains the stated result.

J. W. Green.

Pfluger, A. Konforme Abbildung und eine Verallgemeinerung der Jensenschen Formel. Comment. Math. Helv. 13, 284–292 (1941).

Let $f(z)$ be analytic in a truncated sector, $r_1 < |z| < r_2$, $\varphi_1 < \arg z < \varphi_2$, and continuous and without zeros on the boundary. A single-valued continuous branch of $\arg f(z)$ on the boundary is determined by continuation from $z = r_2 e^{i\varphi_1}$. Let $n(\theta)$ denote the number of zeros of $f(z)$ in $r_1 < |z| < r_2$, $\varphi_1 < \arg z < \theta$, and $n(\rho)$ the number in $r_1 < |z| \leqq \rho$, $\varphi_1 < \arg z < \varphi_2$. Then the author establishes the formulas

$$(1)\quad 2\pi \int_{r_1}^{r_2} n(\rho)\, d\rho/\rho + \int_{r_1}^{r_2} \{\arg f(\rho e^{i\varphi_2}) - \arg f(\rho e^{i\varphi_1})\}\, d\rho/\rho = \int_{\varphi_1}^{\varphi_2} \{\log |f(r_2 e^{i\theta})| - \log |f(r_1 e^{i\theta})|\}\, d\theta,$$

$$(2)\quad 2\pi \int_{\varphi_1}^{\varphi_2} n(\theta)\, d\theta + \int_{r_1}^{r_2} \{\log |f(\rho e^{i\varphi_2})| - \log |f(\rho e^{i\varphi_1})|\}\, d\rho/\rho = \int_{\varphi_1}^{\varphi_2} \{\arg f(r_2 e^{i\theta}) - \arg f(r_1 e^{i\theta})\}\, d\theta.$$

He also considers the modifications necessary when zeros are allowed on the boundary. When $\varphi_1 = 0$, $\varphi_2 = 2\pi$ and $r_1 = 0$, (1) becomes Jensen's formula. In this special case, the terms involving $\log |f|$ drop out of (2), and the resulting formula gives bounds for $\int n(\theta)\, d\theta$ in terms of bounds for $\arg f$. An application of (1) is the theorem: Let $f(z)$ be analytic and bounded in the closed angle $\varphi_1 \leqq \arg z \leqq \varphi_2$. Let

$$\int_1^\infty \{\arg f(\rho e^{i\varphi_2}) - \arg f(\rho e^{i\varphi_1})\}\, d\rho/\rho$$

converge. Then $f(z)$ has no zeros in the angle.

Another application is to the case where $f(z)$ has absolute value 1 on the boundary of the truncated sector and a single zero inside, at a_0; $f(z)$ then maps the region onto the unit circle, and (1) and (2) give information connecting the position of the point a_0 and the relative positions of the four arcs which are the images of the boundary arcs of the region. Thus the author solves for this special region a problem which he states for general regions in the following form. Let the simply-connected region D be bounded by Jordan arcs $J_1, \cdots, J_n$. Let $f(z)$ be analytic in D, continuous and different from zero on its boundary. Required, relations between the zeros of $f(z)$ and the quantities

$$\tilde{\omega}_\nu = \int_{J_\nu} \arg f(z) \cdot |dz/z| \Big/ \int_{J_\nu} |dz/z|,$$

$$\tilde{R}_\nu = \int_{J_\nu} \log |f(z)| \cdot |dz/z| \Big/ \int_{J_\nu} |dz/z|.$$

When $f(z)$ maps D onto the unit circle, $\tilde{\omega}_\nu$ is the "average direction" of the image of J_ν.

R. P. Boas, Jr.

Teichmüller, Oswald. Über Extremalprobleme der konformen Geometrie. Deutsche Math. 6, 50–77 (1941). [MF 5805]

This paper is closely connected in material and spirit with an earlier memoir of the author [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1939, no. 22, 197 pp. (1940); these Rev. 2, 187]. In the former paper he was concerned with conformal mapping of principal domains (Hauptbereiche), that is, oriented finite Riemann manifolds with special points; in this paper, not only do special points map into special points, but also a special class of local parameters is given at each special point (for example, it may be required that a number of derivatives are preserved at a special point). More generally, several principal domains

may be mapped into separate portions of another. An extremal problem of conformal geometry is defined to be a problem of the existence of a conformal mapping of principal domains with given "topologische Festlegung" and "wesentliche Vorschriften" [these concepts are admittedly not precisely defined, though their general character is indicated]. As in the earlier paper, regular quadratic differentials are introduced; a conjectural generalization of the Riemann-Roch theorem involving them is given. With their help a characterization of the extremal mappings for an extremal problem of conformal geometry is formulated. The author gives numerous illustrations of his ideas and methods as applied to special problems, including a derivation of Bieberbach's area theorem and an explanation for the appearance of the transformation $w=1/\sqrt{z}$ in the theory of univalent functions ("Faber's device"). He suggests applications of his ideas to more elaborate coefficient problems for univalent functions. Considerable stress is laid on application of Lie theory to problems of the kind considered in the paper. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Kwesselawa, D. und Vecoua, N. Über ein Randwertproblem der komplexen Funktionentheorie. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 233–240 (1941). (Russian. German summary) [MF 5300]

In this paper the authors give a complete and elementary solution for the special case of the linear Riemann boundary value problem with several unknown functions. Let L be a simple closed smooth curve which lies in the finite part of the complex plane z; let S^+ denote the finite domain bounded by the curve L, and let S^- denote the infinite complementary domain of the infinite z plane. Following N. Muskhelišvili, a function which is regular in the whole plane except at the points of L, and which can be extended continuously over L, is called piece-wise regular. This function is denoted by $\varphi(z)$; the part of $\varphi(z)$ defined in S^+ is denoted by $\varphi^+(z)$ and the part defined in S^- is denoted by $\varphi^-(z)$. The boundary values of $\varphi^+(z)$ and $\varphi^-(z)$ are given by $\varphi^+(x)$ and $\varphi^-(x)$, respectively: $\lim_{z\to x}\varphi^+(z)=\varphi^+(x)$, $\lim_{z\to x}\varphi^-(z)=\varphi^-(x)$.

The nonhomogeneous Riemann boundary value problem in n unknown piece-wise regular functions is formulated as follows: Let there be $n(n+1)$ functions $a_{ik}(x)$, $b_i(x)$ $(i,k=1,\cdots,n)$ defined on the curve L, satisfying the Hölder condition. It is required to find n piece-wise regular functions $\varphi_i(x)$ $(i=1,\cdots,n)$ which satisfy on L the n linear conditions:

$$\text{(A)}\qquad \begin{aligned}\varphi_1^+(x)&=a_{11}(x)\varphi_1^-(x)+\cdots+a_{1n}(x)\varphi_n^-(x)+b_1(x),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ \varphi_n^+(x)&=a_{n1}(x)\varphi_1^-(x)+\cdots+a_{nn}(x)\varphi_n^-(x)+b_n(x).\end{aligned}$$

For the homogeneous case, $b_i(x)=0$. The solution of (A), when $n=1$, is well known to be completely solvable in closed form. The authors also give the complete solutions of the homogeneous and nonhomogeneous problem in integral form. For this it is assumed that, besides satisfying certain natural conditions,

$$\text{(B)}\qquad a_{ik}(x)=\omega_{ik}(x)a_{ik}(x),\qquad i,k=1,\cdots,n,$$

where the $\omega_{ik}(x)$ are the boundary values of functions regular in the region S^+, continuous in the closed region S^++L, and different from zero. *A. Gelbart* (Raleigh, N. C.).

Šaginjan, A. Sur le problème de l'approximation en moyenne dans le domaine complexe. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 285–296 (1941). (Russian. French summary) [MF 5836]

The author continues investigations of O. J. Farrell [Bull. Amer. Math. Soc. **40**, 908–914 (1934)], M. Keldych [Rec. Math. [Mat. Sbornik] **5 (47)**, 391–401 (1939); cf. these Rev. **2**, 80] and himself [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 318–320 (1940); cf. these Rev. **2**, 188] on the completeness of the *SP* (system of polynomials) in the space of regular functions belonging to L_2. Farrell has proved that the *SP* is complete in a Carathéodory domain, that is, such that its complement is connected. The author constructs a domain smaller than, but with a closure equal to, the whole plane in which the *SP* is complete. Further, it is shown that, in $x>0$, $-ae^{-bx^p}<y<ce^{-dx^p}$ $(a,b,c,d>0;\ p<\frac{1}{2})$, the *SP* is not complete. Hence the author infers that the completeness of the *SP* in an unbounded domain depends on metric properties. Applying a linear transformation, he can say that the completeness of rational functions with poles at assigned boundary points depends on metric properties of the domain and the poles. Finally the author finds that the *SP* is complete in two Carathéodory domains with not more than one common boundary point. In the case of two or more common boundary points metric properties have again to be considered. *František Wolf.*

Slobodetzky, L. N. Sur la représentation des fonctions régulières dans le cercle unitaire par certaines séries d'interpolation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 13–15 (1941). [MF 5846]

W. L. and M. K. Gontcharoff investigated expansions of the type $f(z)\sim\sum_{n=0}^{\infty}A_n\Pi_n(z)$, where $f(z)$ is regular in $|z|<1$, and

$$\Pi_0(z)=1;\quad \Pi_n(z)=\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_kz},\quad n\geqq 1;\ |a_k|<1$$

[same C. R. **30**, 298–300 (1941); these Rev. **2**, 355]. Let $|a_k|\leqq|a_{k+1}|$ and let $L(r)$ denote the number of the a_k for which $|a_k|\leqq r$. Let $(1-r)^pL(r)\geqq A>0$, $p>1$, and $\max\log|f(z)|=O((1-r)^{-q})$, $|z|=r$, $q<\frac{1}{2}(p-1)$; then the expansion in question is convergent in $|z|<1$ (uniformly in every closed part). In case $p=1$ certain modifications are necessary; then the convergence is stated only in a part of the unit circle. A more general theorem is also indicated.

G. Szegö (Stanford University, Calif.).

van Veen, S. C. Annäherungsformeln für das vollständige elliptische Integral erster Art in der Nähe von $k=1$. III. Nederl. Akad. Wetensch., Proc. **44**, 825–830 (1941). [MF 5919]

[Parts I, II appeared in the same Proc. **44**, 606–618 (1941); cf. these Rev. **3**, 83.] The approximation obtained in this part of the author's discussion may be written

$$K(k)=\frac{2}{(1+u)^2}\log 2\frac{1+u}{1-u}+\frac{1}{4(1+u)^2}\left(\frac{1-u}{1+u}\right)^4\left[2\log 2\frac{1+u}{1-u}-1+\frac{11\theta}{8}\frac{1-u}{1+u}\right],$$

where $u=k^4$, $|\theta|<1$ and k is assumed to be real with $0\leqq k<1$. This formula may be obtained directly from known

properties of the theta-functions but the author's derivation involves only elementary transformations of the elliptic integral. *M. C. Gray* (New York, N. Y.).

Petersson, Hans. Einheitliche Begründung der Vollständigkeitssätze für die Poincaréschen Reihen von reeller Dimension bei beliebigen Grenzkreisgruppen von erster Art. Abh. Math. Sem. Hansischen Univ. **14**, 22–60 (1941). [MF 5440]

In a former publication [H. Petersson, Math. Ann. **117**, 453–537 (1940); cf. these Rev. **2**, 87] the theory of Poincaré series has been investigated under the following assumptions: Γ is a group of linear substitutions $\tau \to M\tau = (a\tau+b)/(c\tau+d)$, discontinuous in the upper half plane H, with a finite number of generators; moreover Γ contains parabolic substitutions. The author introduces two new types of Poincaré series and proves that the main results of his theory are valid without the last assumption, that is, for arbitrary Fuchsian groups with a finite number of generators. One of the two types of Poincaré series ϕ_{-r} is defined in the following way: Let $v(M)$ be a system of multiplicators of automorphic forms of dimension $-r<-2$, $|v(M)|=1$, τ_0 a point of H, m an integer not less than zero, then

$$\phi_{-r}(\tau, v, \tau_0, \Gamma, m) = \sum_{M\subset\Gamma} \frac{1}{v(M)}\left(\frac{M\tau-\tau_0}{M\tau-\bar{\tau}_0}\right)^m (c\tau+d)^{-r}(M\tau-\bar{\tau}_0)^{-r}.$$

For the other type, τ_0 has to be replaced by a hyperbolic fixed point of Γ. *C. L. Siegel* (Princeton, N. J.).

Stein, Karl. Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderlichen zu vorgegebenen Nullstellenflächen. Math. Ann. **117**, 727–757 (1941). [MF 5654]

Das zweite Cousinsche Problem ist folgendes: Jedem Punkt P eines Bereiches B des Raumes von n komplexen Veränderlichen sei eine Umgebung $U(P)$ und eine dort reguläre Funktion f_P so zugeordnet, dass stets im Durchschnitt zweier Umgebungen $U(P)$ und $U(Q)$ der Quotient f_P/f_Q regulär ist und nicht verschwindet. Gesucht ist eine in B reguläre Funktion F, sodass in jedem $U(P)$ jeweils F/f_P regulär und nicht Null ist. Die Verteilung V der Ortsfunktionen f_P definiert eine in B analytische, $(2n-2)$-dimensionale Mannigfaltigkeit M, Nullstellenmannigfaltigkeit der gesuchten Funktion F. Ein schlichter Bereich B, in dem die zweite Cousinsche Aussage unbeschränkt gilt, ist notwendig ein Regularitätsbereich [B besitzt keine inneren Randpunkte; Thullen, C. R. Acad. Sci. Paris **200**, 720–721 (1935)]. Umgekehrt bewies K. Oka [J. Sci. Hirosima Univ. Ser. A. **9**, 7–19 (1939)], dass in einem Regularitätsbereiche B dann zu einer vorgegebenen analytischen Nullstellenmannigfaltigkeit eine reguläre Funktion existiert, falls es eine stetige Funktion mit diesen Nullstellen gibt. Damit war das zweite Cousinsche Problem auf eine Frage topologischer Natur zurückgeführt, deren Lösung die vorliegende Arbeit gibt. Es gilt: "Zu einer Cousinschen Verteilung V von Ortsfunktionen im endlichblättrigen endlichen Regularitätsbereiche B existiert höchstens dann eine Lösungsfunktion F, wenn die charakteristischen Zahlen von V sämtlich Null sind" (diese sind die Schnittzahlen der Elemente aller 2-dimensionalen Homologiegruppen mod m von B mit M). Sind umgekehrt diese Zahlen Null, so existiert in jedem ganz in B gelegenen Teilbereich eine Lösungsfunktion; ferner gibt es eine in B analytische Funktion, welche dort die vorgeschriebenen Nullstellen, eventuell aber noch weitere besitzt. Entsprechende Sätze gelten bezüglich der Möglichkeit der Darstellung meromorpher Funktionen durch Quotienten regulärer, teilerfremder Funktionen ("Problem von Poincaré," das bekanntlich eng mit dem zweiten Cousinschen Problem zusammenhängt). Sieht man zudem von der Teilerfremdheit der Quotienten ab, so gilt: Ist die Funktion Φ im endlichen, endlichblättrigen Regularitätsbereich B meromorph und sind die charakteristischen Zahlen der durch Φ definierten Polflächen sämtlich Null, so ist Φ in B als Quotient regulärer Funktionen darstellbar. In Zylinderbereichen sind die Verhältnisse besonders einfach: Jede in einem Zylinderbereiche vorgegebene Cousinsche Verteilung lässt sich so ergänzen, dass die neue Verteilung eine eindeutige Lösungsfunktion gestattet. Ferner: Jede in einem Zylinderbereiche meromorphe Funktion lässt sich dort als Quotient regulärer (nicht notwendig teilerfremder) Funktionen darstellen. Als letzte Anwendung gibt Verfasser die Konstruktion schlichter Regularitätsbereiche an, zu denen es reguläre Funktionen gibt, die sich nicht in dort gleichmässig konvergente Folgen rationaler Funktionen entwickeln lassen. [Vgl. auch Stein, S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1939**, 139–149; diese Rev. **1**, 215; ferner K. Oka, Jap. J. Math. **17**, 517–531 (1941); diese Rev. **3**, 85.] *P. Thullen* (Quito).

Bergman, Stefan and Spencer, D. C. On distortion in pseudo-conformal mapping. Trans. Amer. Math. Soc. **51**, 133–163 (1942). [MF 6092]

Die im Einheitsdizylinder $\mathfrak{B}_1(|z_1|<1, |z_2|<1)$ regulären Funktionen zweier komplexen Veränderlichen:

$$w_1(z_1, z_2) = a^{(1)}_{1,0}z_1 + z_1^2 z_2 \cdot f_1(z_1, z_2);$$
$$w_2(z_1, z_2) = a^{(2)}_{0,1}z_2 + z_1 z_2^2 \cdot f_2(z_1, z_2),$$

$|a^{(1)}_{1,0}| = |a^{(2)}_{0,1}| = 1$, mögen $\mathfrak{B}_1$ in einen (nicht notwendig schlichten) Bereich $\mathfrak{B}_2$ abbilden. Ferner sei $\mathfrak{B}_2^\dagger$ der grösste in $\mathfrak{B}_2$ liegende Teilbereich mit der Eigenschaft, dass jedes Ebenenstück: $\arg w_k = \psi_k$ $(k=1, 2)$ ein zusammenhängendes Gebiet mit $(0, 0) = (w_1(0, 0), w_2(0, 0))$ als Randpunkt ausschneidet (im Falle einer komplexen Veränderlichen ist der entsprechend gebildete Bereich ein Sternbereich). Es wird dann eine Ungleichung für den Volumeninhalt von $\mathfrak{B}_2^\dagger$ abgeleitet, die von den Werten der Ableitungen erster Ordnung der Abbildungsfunktionen $w_k(z_1, z_2)$ in $\mathfrak{B}_1$ und im besonderen von der "mittleren Wertigkeit" (mean valency) des Bereiches $\mathfrak{B}_1^\dagger$ abhängt. (Diese Ungleichung ist eine Verallgemeinerung eines Satzes von Golusin [Rec. Math. [Mat. Sbornik] N.S. **2** (**44**), 617–619 (1937)] über Abbildungen des Einheitskreises durch reguläre Funktionen einer Veränderlichen, nach dem bei entsprechenden Bedingungen $A[w^{-1}(\mathfrak{B}_2^\dagger)] \cdot A(\mathfrak{B}_2^\dagger) \geqq \pi^3$, wobei unter $A(\mathfrak{B})$ der Flächeninhalt von $\mathfrak{B}$ zu verstehen ist.) Der hier fundamentale Begriff der mittleren Wertigkeit eines Bereiches $\mathfrak{B}$ wird gewonnen, indem man unter Zugrundelegung eines geeigneten Koordinatensystems den Volumeninhalt von $\mathfrak{B}$ mit dem eines Grundbereiches $\mathfrak{B}_0$ vergleicht (bei nicht schlichtem Bereich $\mathfrak{B}$ ist jeder Punkt des Raumes so oft zu zählen als er vorkommt). Als Grundbereich wird in der vorliegenden Arbeit die aus zwei aneinanderstossenden Dizylindern zusammengesetzte Menge $(\mathfrak{B}_0)$ $|w_1| \leqq 1$, $|w_2| \leqq 1$; und $|w_1| \geqq 1$, $|w_2| \geqq 1$ gewählt. Die mittlere Wertigkeit eines Bereiches $\mathfrak{B}$ hängt wesentlich von der Wahl des Bezugspunktes und des Grundbereiches $\mathfrak{B}_0$ ab.

P. Thullen (Quito).

Functional Analysis, Ergodic Theory

Dieudonné, Jean. Topologies faibles dans les espaces vectoriels. C. R. Acad. Sci. Paris 211, 94–97 (1940). [MF 5337]

Dieudonné, Jean. Équations linéaires dans les espaces normés. C. R. Acad. Sci. Paris 211, 129–131 (1940). [MF 5340]

If E and E' are any two vector spaces, and $B(x, x')$ is a numerical bilinear form which vanishes for all points in one space if and only if the point of the other space is the origin, then for any points $x_1', \cdots, x_n'$ of E' the inequalities $|B(x, x_m')| \leqq 1$, $m=1, \cdots, n$, define a neighborhood topology in E. In particular, if E has an initial topology and E' is the space of continuous distributive forms, the new topology is "weak" in relation to the initial "strong" topology. The author announces many properties. The following theorem is remarkable. If the initial topology of E is von Neumann's topology but without countability restriction, then it is derivable from a norm if and only if there exists a neighborhood of the origin which is weakly bounded. *S. Bochner* (Princeton, N. J.).

Šmulian, V. L. Sur les espaces linéaires topologiques. II. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 727–730 (1941). (Russian. French summary) [MF 5507]

The author extends a theorem given in the first part [Rec. Math. [Mat. Sbornik] N.S. 7 (49), 425–448 (1940), in particular, p. 427; these Rev. 2, 102]. We state the whole theorem. Let Γ be a total, linear set of functionals on a linear topological space E, and let $E(\Gamma)$ be the set E topologized by the neighborhoods of the origin: $U(f_1, \cdots, f_n; \epsilon) = E_x[|f_i(x)| < \epsilon,\ i=1, \cdots, n;\ f_i \in \Gamma]$. Then, if K is a convex bounded set of E, the following statements are equivalent: (1) K is bicompact in $E(\Gamma)$. (2) Every linear functional $\Phi(f)$ defined on Γ and continuous in the norm $\|f\|_K = \sup_{x \in K} |f(x)|$, and having the property $\inf_{x \in K} f(x) \leqq \Phi(f) \leqq \sup_{x \in K} f(x)$ for $f \in \Gamma$ has the form $\Phi(f) = f(x)$ for some $x \in K$. (3) If $K_1 \supset K_2 \supset \cdots \supset K_\xi \supset \cdots$ is a transfinite sequence of convex, Γ-separable (that is, $x_0 \in K_\xi$ implies the existence of $f_0 \in \Gamma$ such that $f_0(x_0) > \sup_{x \in K_\xi} f_0(x)$) sets such that $K \cdot K_\xi \neq 0$ for any ξ, then $K \cdot K_1 \cdot K_2 \cdots K_\xi \cdots \neq 0$. (4) K is transfinitely closed with respect to functionals in Γ. (5) For each transfinite sequence $\{x_\xi\} \subset K$, there exists $x_0 \in K$ such that $\lim f(x_\xi) = f(x_0)$ for each $f \in \Gamma$ for which the limit on the left exists. These statements imply (a) K is Γ-separable and (b) K is closed in $E(\Gamma)$, (a) and (b) being equivalent. If every totally bounded subset of $E(\Gamma)$ is complete in the sense of directed sets, (a) and (b) are equivalent to the first five statements. The results are applied to Banach spaces to give known theorems. *J. V. Wehausen.*

Šmulian, V. L. Sur la structure de la sphère unitaire dans l'espace de Banach. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 545–561 (1941). (French. Russian summary) [MF 5500]

Conditions are determined for differentiability (in various senses) of the norm in the space $M(Q)$ of bounded functions on a set Q, the spaces (m), (c), (c_0), the space $C(S)$ of bounded continuous functions on a normal Hausdorff space S and the space L_σ of functions on the unit interval with $\int_0^1 |x(t)|\, d\sigma(t) < \infty$, where $\sigma(t)$ is a monotone function. The results for $M(Q)$ are applied to arbitrary Banach spaces to obtain conditions for reflexiveness in terms of strong differentiability. Reflexiveness is also related to the weak completeness of "frontier sets" (partially ordered sets $\{x_\alpha\}$ such that $\|x_\alpha\| \leqq 1$ and $\lim \|(x_\alpha' + x_\alpha'')/2\| = 1$). Normed rings with unit element are considered and strong differentiability of the norm in $C(S)$ is related to the connectedness of S. Strong differentiability is also used to characterize (l) and (L) in the class L, and (c_0), (c) and subspaces of (m) containing (c) in a certain class of normed rings. *J. V. Wehausen* (Columbia, Mo.).

Day, Mahlon M. A property of Banach spaces. Duke Math. J. 8, 763–770 (1941). [MF 5959]

The author proves that various types of weak completeness are necessary and sufficient for the reflexivity of a Banach space, the essential tool being Helly's theorem on linear functionals. A list of criteria is given which are necessary and sufficient for the reflexivity of every λ-separable subspace of a Banach space B, and it is shown that B has this property if and only if every isomorphic image of B has it. *J. A. Clarkson* (Philadelphia, Pa.).

Sobczyk, Andrew. Projection of the space (m) on its subspace (c_0). Bull. Amer. Math. Soc. 47, 938–947 (1941). [MF 5942]

The basic result of this paper is that if (B) is a separable space such that $(c_0) \subset (B) \subset (m)$, then there is a projection of (B) on (c_0) of norm 2. A previous result of Phillips that there is no projection of (m) on (c) [Trans. Amer. Math. Soc. 48, 516–541 (1940), in particular, p. 539; cf. these Rev. 2, 318] implies that there is none of (m) on (c_0) and hence none of (m) on (B). The paper also contains a number of remarks on an extension theorem of Phillips for linear transformations to (m). *F. J. Murray* (New York, N. Y.).

Kakutani, Shizuo. Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.) Ann. of Math. (2) 42, 994–1024 (1941). [MF 5527]

Among the class of Banach lattices the space $C(\Omega)$ of continuous functions on a compact Hausdorff space Ω, as well as its subspaces which are lattice as well as metrically closed, are characterized in terms of order. A Banach lattice is a Banach space and a linear vector lattice in which the relation $\geqq$ is continuous. If in addition it is assumed that $x \wedge y = 0$ implies $\|x+y\| = \|x-y\|$ and that $x \geqq 0$, $y \geqq 0$ imply $\|x \vee y\| = \max(\|x\|, \|y\|)$ the Banach lattice is called an abstract M space. For every such space (AM) there is a compact Hausdorff space Ω and points $t_\alpha, t_\alpha' \in \Omega$ and real λ_α with $0 \leqq \lambda_\alpha < 1$ such that (AM) is isometric and lattice isomorphic to the subspace of $C(\Omega)$ consisting of those $x \in C(\Omega)$ for which $x(t_\alpha) = \lambda_\alpha x(t_\alpha')$. If (AM) has a unit 1 in the sense that $1 \geqq 0$, $\|1\| = 1$, $x \leqq 1$ providing $\|x\| \leqq 1$, then (AM) is isometric and lattice isomorphic to all of some $C(\Omega)$, where Ω is a compact Hausdorff space. This latter result was also proved by M. and S. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 427–430 (1940); cf. these Rev. 2, 222]. The method of proof of these two fundamental theorems consists in first showing that distinct elements $x, y \in (AM)$ may be distinguished by a continuous homomorphism of (AM) onto the reals. This gives the algebraic embedding of (AM) in a direct product of the reals. The topological problem is then solved by use of the Tychonoff theorem on the product of compact spaces. Among the many applications of these theorems we mention the extension theorem of E. Čech which asserts that a completely regular Hausdorff space Ω' may be embedded in a compact Hausdorff space Ω in such

a way that Ω' is dense in Ω and every bounded continuous function on Ω' has a unique continuous extension to Ω. It is also shown that the linear functional on $C(\Omega)$ is a Lebesgue integral with respect to a finite completely additive measure defined for Borel sets in Ω, a result which shows that the conjugate of an abstract M space is an abstract L space. The representation theory here presented should be compared with that of M. H. Stone [Proc. Nat. Acad. Sci. U. S. A. **26**, 280–283 (1940); cf. these Rev. **1**, 338] who characterizes $C(\Omega)$ in terms of order and its ring properties.
N. Dunford (New Haven, Conn.).

Bohnenblust, H. F. and Kakutani, S. Concrete representation of (M)-spaces. Ann. of Math. (2) **42**, 1025–1028 (1941). [MF 5528]

Replacing the condition (1): "$0 \leqq x$, $0 \leqq y$ implies $\|x \vee y\| = \max(\|x\|, \|y\|)$" of the preceding review by the weaker condition (2): "$x \wedge y = 0$ implies $\|x \vee y\| = \max(\|x\|, \|y\|)$," the authors show that the lattice is still isomorphic and isometric with a subspace of some $C(\Omega)$. The proof that (2) implies (1) is not direct but made via the representation.
N. Dunford (New Haven, Conn.).

Botts, Truman. On convex sets in linear normed spaces. Bull. Amer. Math. Soc. **48**, 150–152 (1942). [MF 6195]

The author gives a new proof of the theorem: In a linear normed space two convex bodies (that is, convex sets with inner points) having no common inner points are separated by a plane. Previous proofs were given by M. Eidelheit [Studia Math. **6**, 104–111 (1936)] and S. Kakutani [Proc. Imp. Acad. Tokyo **13**, 93–94 (1937)]. In the present proof it is shown by elementary considerations that a convex cone exists which contains one of the convex bodies, and which is such that any one of its supporting planes separates the two convex bodies. The proof uses a theorem of Mazur: Through every boundary point of a convex body there passes a plane supporting the body [Studia Math. **4**, 70–84 (1933), in particular p. 74].
G. B. Price.

Michal, Aristotle D., Davis, Roderick and Wyman, Max. Polygenic functions in general analysis. Ann. Scuola Norm. Super. Pisa (2) **9**, 97–107 (1940). [MF 5456]

The authors consider a real Hilbert space E (separable or non-separable), and form from it a complex Banach space $E(C)$ of pairs $z = (x, y) = x + iy$ of elements of E. A function $f(z) = f_1(x, y) + if_2(x, y)$ defined in a domain D of $E(C)$, having its values in a complex couple space of the same character as $E(C)$, is said to be polygenic in D if f_1, f_2 are continuous in D, and if

$$\lim_{t \to 0} \frac{f_i(x + t\xi, y + t\eta) - f_i(x, y)}{t}, \qquad i = 1, 2,$$

exists and defines a continuous function of (x, y) in D for each pair ξ, η in E. In case E is the space of real numbers this reduces to the demand that f_1, f_2 admit continuous first partial derivatives in D.

A polygenic function possesses a directional Gateaux differential

$$\delta f(z; \Delta z, \phi) = \lim_{r \to 0} \frac{f(z + re^{i\phi}\Delta z) - f(z)}{re^{i\phi}}$$

which is continuous in D for each Δz in $E(C)$ and each real ϕ. It can be decomposed into two parts:

$$\delta f(z; \Delta z, \phi) = D(f(z), \Delta z) + e^{-2i\phi} P(f(z), \Delta z).$$

f is analytic in the sense of A. E. Taylor [Ann. Scuola Norm. Super. Pisa (2) **6**, 277–291 (1937)] if and only if $P(f(z), \Delta z) \equiv 0$. Furthermore, $D(f(z), \Delta z) \equiv 0$ implies that f is an analytic function of $x - iy$. A theorem is obtained which characterizes $D(f(z), \Delta z)$ and $P(f(z), \Delta z)$ for all polygenic functions such that f_1, f_2 possess continuous second partial Fréchet differentials.
A. E. Taylor.

Bochner, S. Hilbert distances and positive definite functions. Ann. of Math. (2) **42**, 647–656 (1941). [MF 4963]

Let $\mathfrak{S}$ be a separable and compact space with a Lebesgue measure possessing the usual properties, and $\mathfrak{C}$ a transitive group of motions $s: P \to sP$ in $\mathfrak{S}$. A function $F(P, Q, \cdots)$ in $\mathfrak{S}$ is called group invariant if $F(sP, sQ, \cdots) \equiv F(P, Q, \cdots)$ for all $s \in \mathfrak{C}$. The main subject of this paper is the study of those group-invariant metrics $\rho(P, Q)$ in $\mathfrak{S}$ (generating the given topology of $\mathfrak{S}$, possibly with identifications) with which $\mathfrak{S}$ can be imbedded isometrically into Hilbert space, that is, the Hilbert distances in $\mathfrak{S}$. Following K. Menger [Math. Ann. **103**, 466–501 (1930)] and I. J. Schoenberg [Ann. of Math. (2) **41**, 715–726 (1940); cf. these Rev. **2**, 130], a connection is established between these distances and the positive functions of $\mathfrak{S}$, that is, those continuous and group-invariant functions $f(P, Q)$ for which

$$\sum_{i,j=1}^{n} f(P_i, P_j)\rho_i\bar{\rho}_j \geqq 0$$

for all finite systems P_i, ρ_i, $i = 1, \cdots, n$. Then $\rho(P, Q)$ turns out to be a Hilbert distance if and only if it is of the form $(C - f(P, Q))^{\frac{1}{2}}$, $f(P, Q)$ being any positive function in $\mathfrak{S}$ (C is the constant value of $f(\phi, \phi)$).

The representation theory of $\mathfrak{C}$ in $\mathfrak{S}$ following Bochner, J. v. Neumann and H. Weyl is carefully gone into, and explicit forms for $\rho^2(P, Q)$-s orthogonal expansions in representation functions of $\mathfrak{C}$ in $\mathfrak{S}$ are obtained. They turn out to be absolutely and uniformly convergent expressions of the absolute value square sum type.

[For the analogous problem of determining the Hilbert distances in the case of (non-compact) vector groups, cf. the theory of screw lines, J. von Neumann and I. J. Schoenberg, Trans. Amer. Math. Soc. **50**, 226–251 (1941); cf. these Rev. **3**, 37. For the general theory of positive functions on (non-compact) commutative groups, cf. A. Powzner, C. R. (Doklady) Acad. Sci. USSR (N.S.) **28**, 294–295 (1940), and A. Raikow, C. R. (Doklady) Acad. Sci. USSR (N.S.) **27**, 324–327 (1940); **28**, 296–300 (1940); cf. these Rev. **2**, 223.]
J. von Neumann (Princeton, N. J.).

Ridder, J. Mass- und Integrationstheorie in Strukturen. Acta Math. **73**, 131–173 (1941). [MF 6098]

A. Rosenthal [Nachr. Ges. Wiss. Göttingen **1916**, 305–321] has formulated a system of axioms for regular outer measure which is equivalent to systems formulated by Carathéodory [Nachr. Ges. Wiss. Göttingen **1914**, 404–420; Vorlesungen über reelle Funktionen, Teubner, Leipzig, 1927, Chapters 5 and 6]. Carathéodory defines regular inner measure in terms of regular outer measure, but Rosenthal's characterization of regular inner measure is carried out independently by means of a system of axioms similar to those for outer measure. The author of the present paper characterizes outer and inner measure by systems of axioms for which he claims advantages over the Rosenthal systems: Axioms I–IV and V for outer, and axioms I–IV and V′ for inner, measure lead to bounded additive measure functions; all properties derivable from axioms I–IV hold for outer and for inner measures; to each of the properties

derivable from axioms I–IV and V there is a corresponding property derivable from axioms I–IV and V′; to each outer (inner) measure is adjoined an inner (outer) measure in such a way that they both lead to the same field of measurable sets, with equal measure numbers: Axioms I–VII for outer, and axioms I–IV, V′, VI and VII′ for inner, measure lead to completely additive measures; all properties derivable from axioms I–VI hold for both measures; to each property derivable from axioms I–VII for outer measure there is a corresponding property derivable from axioms I–IV, V′, VI, VII′ for inner measure; to each outer (inner) measure is adjoined an inner (outer) measure in such a way that both measures lead to the same σ-field of measurable sets, with equal measure numbers. [Sets measurable by a Carathéodory inner measure satisfying his first two axioms form a field, but even if the inner measure function satisfies all five axioms the corresponding measurable sets do not form a σ-field.]

These measure representations are not restricted to subsets of Euclidean space, or of abstract metric spaces, but apply also to general structures whose elements have been called somen by Carathéodory. The paper concludes with definitions of Riemann-Stieltjes and Lebesgue-Stieltjes integrals of abstract functions defined on somen.

R. L. Jeffery (Wolfville, N. S.).

Dunford, Nelson and Tamarkin, J. D. A principle of Jessen and general Fubini theorems. Duke Math. J. **8**, 743–749 (1941). [MF 5957]

The first part of this paper extends the theorem of Jessen [Acta Math. **63**, 249–323 (1934), in particular, p. 270] that a real measurable function on a direct product space which is independent of any change of a finite number of coordinates is necessarily almost everywhere a constant, to a more general setting. For this purpose the real product space of denumerably infinite dimension is replaced by S, a product of arbitrary spaces S_α, the subscripts α ranging over a general set A, each S_α containing a Borel field $\mathfrak{F}_\alpha$ of sets which includes S_α. These give rise to elementary sets $E_\sigma = \Pi_{\alpha\in\sigma}[E_\alpha] \times \Pi_{\beta\in\bar{\sigma}} S_\beta$, where σ is a finite subset of A, $\bar{\sigma}$ its complement relative to A, and $E_\alpha \in \mathfrak{F}_\alpha$. These in turn are the basis of the Borel field $\mathfrak{F}$ in S. If R is a Hausdorff ring with only two idempotents, then absolutely additive set functions φ_α on $\mathfrak{F}_\alpha$ to R with $\varphi_\alpha(S_\alpha)=1$ generate an absolutely additive set function φ on $\mathfrak{F}$ to R such that

$$\varphi[\prod_{\alpha\in\sigma} E_\sigma \times \prod_{\beta\in\bar{\sigma}} S_\beta] = \prod_{\alpha\in\sigma} \varphi_\alpha(E_\alpha)$$

if $E_\alpha \in \mathfrak{F}_\alpha$. For sets M of $\mathfrak{F}$ having the Jessen property J of containing with any element $x(\alpha)$ all elements $y(\alpha)$ differing from $x(\alpha)$ at a finite number of α, $\varphi(M)$ is idempotent. The Jessen theorem then affects a function f on S to a separable metric space E, measurable in the sense that, for every Borel set D of E, $f^{-1}(D) \in \mathfrak{F}$, and states if f has the property J, that is, for any q of E, $f^{-1}(q)$ has property J, or f is independent of change of a finite number of coordinates, then f is a constant except on a set where $\varphi=0$.

The second part of the paper assumes f on S to a Banach space E, and φ a positive valued function. Then we have available the spaces of integrable functions: $L_p(E)$, $1 \leqq p < \infty$ and $L_p^B(E)$, where B is any subset of A with corresponding norms. If $f^B(x) = \int_{S_{\bar{\beta}}} f d\varphi^{\bar{B}}$ ($\bar{B}$ the complement of B relative to A), then $\lim_\sigma f^\sigma(x) = f(x)$, while $\lim_\sigma f^{\bar{\sigma}}(x) = \int_S f d\varphi$, where σ are finite subsets of A, the limit is the Moore limit and taken in the sense of norms on $L_p(E)$, generalizing corresponding theorems of Jessen [loc. cit., p. 279]. In case the range A is denumerable, the convergence in $L(E)$ in the theorem is also almost everywhere, if taken sequentially.

T. H. Hildebrandt (Ann Arbor, Mich.).

Alexandroff, A. D. Additive set-functions in abstract spaces. Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 563–628 (1941). (English. Russian summary) [MF 5501]

These are chapters 2 and 3 of a monograph under the same title to comprise six chapters in all. In chapter 1, which appeared in Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 307–348 (1940) [these Rev. **2**, 315], the notations were explained and the plan of each chapter carefully mapped out.

Chapter 2 is titled "Linear functionals and charges." The main result is that in a normal space for every linear functional $L(f)$ of the continuous bounded functions on this space there exists one and only one "charge" $\mu(E)$ such that $L(f) = \int f(x)\mu(dE)$; the integral is a Lebesgue-Radon integral extended over the space and a "charge" is a sort of pseudo-measure (it need be neither totally additive nor non-negative; see chapter 1). The proof proceeds by first demonstrating that if $L(f)$ is non-negative then one obtains the charge μ for closed sets F in setting $\mu(F) = \inf L(f)$, the infimum being taken over the family of functions f which are continuous, bounded, non-negative, and not less than 1 on F. The proof is completed by showing that each linear functional and charge can be decomposed into the difference of two non-negative parts. The author remarks that this process is related to work of A. Markoff [Rec. Math. [Mat. Sbornik] N.S. **4** (**46**), 165–190 (1938)], which is in turn connected with von Neumann's paper, "Zum Haarschen Mass in topologischen Gruppen" [Compositio Math. **1**, 106–114 (1934)].

Chapter 3 is called "Totally additive and real charges." It is shown that in a normal space every charge is totally additive if and only if the space is compact, a charge is totally additive if its associated integral is continuous in the sense of bounded convergence, and if a space is topologically imbeddable in a certain way then every totally additive charge is concentrated on a countable family of compact sets. Finally there is introduced a concept of "real" charge, more restrictive than that of total additivity. Results are then obtained for spaces which are not normal but merely (Tychonoff) totally-regular. *H. Wallman.*

Goldstine, H. H. Linear functionals and integrals in abstract spaces. Bull. Amer. Math. Soc. **47**, 615–620 (1941). [MF 5055]

It is shown that the Daniell integral and the integral similarly defined by Banach in his addendum to Saks's "Theory of the Integral" are Lebesgue integrals with respect to regular Carathéodory outer measures.

N. Dunford (New Haven, Conn.).

Artemenko, A. On positive linear functionals in the space of almost periodic functions of H. Bohr. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 111–114 (1940). (Russian. English summary) [MF 4737]

Let $\mathfrak{B}$ be the linear space of almost periodic functions of Bohr with the norm $\|x\| = \sup |x(t)|$. The author proves the following theorem: In order that there exist a linear positive functional $f(x)$ such that $f(e^{\Omega t}) = \psi(\lambda)$, $-\infty < \lambda < \infty$,

it is necessary and sufficient that

$$\sum_{k,l=1}^{n} \psi(\lambda_k-\lambda_l)\xi_k\bar{\xi}_l \geqq 0$$

for arbitrary choice of complex $\xi_1, \xi_2, \cdots$. The functional $f(x)$ is determined by $\psi(\lambda)$ uniquely. The proof is based on the following lemma: Given a non-negative polynomial $x(t)=\sum_{p=1}^{n} a_p e^{\lambda_p t}$, $\epsilon>0$, there exists a real polynomial $z(t)$ such that $\rho(x-z^2)<\epsilon$. Here $\rho(y)=\sum_{l=1}^{k}|b_l|$ whenever $y=\sum_{l=1}^{k} b_l e^{\lambda_l t}$. *J. D. Tamarkin* (Providence, R. I.).

Bochner, S. On a theorem of Tannaka and Krein. Ann. of Math. (2) **43**, 56–58 (1942). [MF 6059]

Krein has shown [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 9–12 (1941); these Rev. **2**, 316] that an additive functional L defined on the class C_0 of all finite linear combinations of irreducible representation coefficients of an arbitrary group G will necessarily be positive and bounded and uniquely extensible to the class C of all almost periodic functions on G if $L(|g|^2)\geqq 0$ for all $g\varepsilon C_0$. This theorem (which includes an earlier theorem of Tannaka) was proved by the methods of I. Gelfond and N. Wiener, which involve difficult concepts apparently extraneous to the problem. Bochner gives a direct and relatively simple proof which stays within the technique of uniform approximation.

R. H. Cameron (Cambridge, Mass.).

Soboleff, V. I. Sur les éléments caractéristiques de certains opérateurs non linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 735–737 (1941). [MF 5240]

Let $f(x)$ be a functional defined for all x in real (separable) Hilbert space H, and possessing a Fréchet differential $df(x, h)$ for all $x\varepsilon H$. Since, for each x, $df(x, h)$ is bounded linear in h, it has the form (x^*, h), and $f(x)$ defines a transformation A, $Ax=x^*$. Such an operator A is called symmetric. The author shows that, if A is in addition totally continuous and positive $((Ax, x)>0$ for all $x\neq 0)$, and satisfies $A(-x)=-Ax$, $\|Ax-Ay\|=M\|x-y\|$, for all x and y in H, then A has infinitely many characteristic elements of norm unity, the corresponding characteristic numbers forming a monotone non-increasing sequence with limit zero. The writer also points out that an analogous result holds for operators from $L^{(p)}(0, 1)$ to $L^{(q)}(0, 1)$, $1<p<2$, $1/p+1/q=2$, provided a characteristic element is defined as a solution of $Ax=\lambda|x|^{p-1}$ sign x. The paper includes significant examples of integral equations to which the theory applies. *J. W. Calkin* (Chicago, Ill.).

Calkin, J. W. Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. Ann. of Math. (2) **42**, 839–873 (1941). [MF 5520]

The set $\mathfrak{B}$ of all bounded operators on a Hilbert space possesses non-trivial two sided ideals, for example, $\mathfrak{T}$, the set of totally continuous transformations, and $\mathfrak{F}$, the set of transformations with a finite dimensional range. If $\mathfrak{g}$ is a two sided ideal and not $\mathfrak{B}$ or $\{0\}$, then $\mathfrak{F}\subset\mathfrak{g}\subset\mathfrak{T}$. For $T\varepsilon\mathfrak{g}\subset\mathfrak{T}$, $|T|=(T^*T)^{\frac{1}{2}}$ has a pure point spectrum, given by a sequence $\{\lambda_i\}$ with $\lambda_n\to 0$ $(\lambda_n\geqq 0)$. For such an $\mathfrak{g}$, the set $\mathfrak{z}$ of such sequences $\{\lambda_n\}$ is additive, invariant under permutations of the indices and contains any sequence which is majorized by any member sequence. Conversely, it is shown that, if $\mathfrak{z}$ is a set of sequences having the above mentioned properties, there is a two sided ideal $\mathfrak{g}$ for which it is the corresponding $\mathfrak{z}$. [This relation between $\mathfrak{g}$ and $\mathfrak{z}$ is due to J. v. Neumann.]

Let $\mathfrak{B}/\mathfrak{g}$ denote the set of residue classes of elements of $\mathfrak{B}$ modulo $\mathfrak{g}$. For α and β in $\mathfrak{B}/\mathfrak{g}$, the operations $\alpha+\beta$, $\alpha\beta$, $a\cdot\alpha$ and α^* can be defined in the usual way for residue classes. Thus α^* is the class of adjoints of elements of α. If $\alpha=\alpha^*$, α contains a self-adjoint A. If $\epsilon^2=\epsilon$, $\epsilon=\epsilon^*$, ϵ contains a projection E; ϵ is then termed an idempotent. When $\mathfrak{g}$ is closed under the operation of extracting the square root of a definite operator, $\omega^*\omega=\epsilon$, an idempotent, implies that ω contains a partially isometric W and $\omega^*\omega=1=\omega\omega^*$ implies that ω contains a partially isometric W with finite deficiency indices. The center of $\mathfrak{B}/\mathfrak{g}$ is $\{a\cdot 1\}$. For $\mathfrak{g}=\mathfrak{T}$ and $\alpha\varepsilon\mathfrak{B}/\mathfrak{T}$, let $|\alpha|=$g. l.b. of the bounds of the $A\varepsilon\alpha$. Relative to this norm $\mathfrak{B}/\mathfrak{T}$ is a complete metric non-separable space.

To show that $A_1-A_2\varepsilon\mathfrak{T}$ implies $|A_1|-|A_2|\varepsilon\mathfrak{T}$ a new device is introduced. Using a generalization of the notion of limit, the set of sequences of elements of $\mathfrak{H}$ weakly convergent to zero is shown to possess an inner product. If one identifies two sequences for which the difference of corresponding elements approaches zero strongly in the generalized sense of the limit, one obtains a space $\mathfrak{L}'$, which can be completed into a space $\mathfrak{L}$ of dimension c. To each $\alpha\varepsilon\mathfrak{B}/\mathfrak{T}$, there is a bounded transformation $T(\alpha)$ on $\mathfrak{L}$. The bound of $T(\alpha)$ is $|\alpha|$ and the relations $+$, $a\cdot$, $\cdot$, and * are preserved by this correspondence. However the set of $T(\alpha)$ is not closed with respect to strongly convergent sequences. If $A\varepsilon\alpha$ is self-adjoint, $T(A)=T(\alpha)$ is also. The spectrum of A and that of $T(A)$ are related as follows. Let $E(\lambda)$ denote the resolution for A. If λ is such that there is a λ_1 and λ_2 such that $\lambda_1<\lambda<\lambda_2$ and $E(\lambda_2)-E(\lambda_1)$ has a finite dimensional range, then λ is in the resolvent set for $T(A)$. Otherwise λ is in the point spectrum of $T(A)$ with c multiplicity. In the first case, λ is said to be in the augmented resolvent set for A; in the second case, it is in the condensed spectrum of A. There is a generalization of these spectral results to non-self-adjoint elements of $\mathfrak{B}$. On $\mathfrak{L}$, if $F(\lambda)=\lim_{\epsilon\to 0} T(E(\lambda+\epsilon^2))$, $F(\lambda)$ is the resolution for $T(A)$. Consider now the resolutions $E_1(\lambda)$ and $E_2(\lambda)$ of two self-adjoint elements A_1 and A_2 such that $A_1-A_2\varepsilon\mathfrak{T}$. For these, if $\mu<\lambda$, there is a $T_{\mu,\lambda}\varepsilon\mathfrak{T}$, such that $E_1(\mu)+T_{\mu,\lambda}\leqq E_2(\lambda)$. For $\mu=\lambda$, equality is obtainable if μ is in the augmented resolvent set of A_1. The author indicates a generalization of the above theory in which $\mathfrak{B}$ is replaced by a factor of class II_∞ and also a generalization to spaces of dimensionality greater than $\aleph_0$. *F. J. Murray.*

Julia, Gaston. Sur une décomposition en produit infini des opérateurs linéaires de l'espace hilbertien. C. R. Acad. Sci. Paris **212**, 829–831 (1941). [MF 5030]

In a previous note [C. R. Acad. Sci. Paris **212**, 733–736 (1941); these Rev. **3**, 51], the author showed the existence of a linear operator A defined in a Hilbert space H and taking on a given set of values A_n $(n=1, 2, \cdots)$ in H for values e_n which form a complete orthonormal set in H, that is, $Ae_n=A_n$ $(n=1, 2, \cdots)$. In the present paper it is shown that A is expressible in the form $\Pi_{p=1}^{\infty}\mathfrak{A}_p U$, where U is unitary, the operators A_p are of a certain canonical type and the product converges strongly. To exhibit the character of the operator $\mathfrak{A}_p$ let the elements A_n be transformed by the well-known Schmidt process into an orthonormal set ϵ_n such that $A_n=\sum_{i=1}^{n}\alpha_{ni}\epsilon_i$ $(n=1, 2, \cdots)$. Then the operator $\mathfrak{A}_p$ $(p=1, 2, \cdots)$ is characterized by the conditions that $\mathfrak{A}_p e_p=A_p$, $\mathfrak{A}_p e_n=e_n$ $(n\neq p)$. It is a bounded operator and, if $x=\{x_i\}$ is in H, then $\mathfrak{A}_p x=x+x_p(A_p-e_p)$.

The point spectrum of each $\mathfrak{A}_p$ is made up of the values 1 and α_{pp}, the former being of infinite order and the latter of order 1. *H. H. Goldstine* (Ann Arbor, Mich.).

Neumark, M. Direkte Polynome von symmetrischen Operatoren und ihre selbst-adjungierte Fortsetzungen. I. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 629–666 (1941). (German. Russian summary) [MF 5502]

The author continues his researches in the theory of symmetric transformations in Hilbert space [see, in particular, Bull. Acad. Sci. URSS 4, 53–104, 277–318 (1940); cf. these Rev. 2, 104, 105] with a detailed analysis of direct polynomials of two symmetric operators, one of which is self-adjoint, and the self-adjoint extensions of such polynomials. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces, $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ their direct product. Then if A_1 and A_2 are linear operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and $p(\xi, \eta) = \sum_{k+l \leq n} a_{kl} \xi^k \eta^l$ a polynomial with complex coefficients, it is possible under suitable restrictions on A_1 and A_2 to give the formal symbol $p(A_1, A_2)$ significance as an operator in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ by means of the equation

$$(1) \qquad p(A_1, A_2)(f \otimes g) = \sum_{k+l \leq n} a_{kl}(A_1^k f \otimes A_2^l g)$$

and the usual process of linear extension. In particular, the right member of (1) has meaning for $f \in \mathfrak{Q}(A_1^{\nu_1})$, $g \in \mathfrak{Q}(A_2^{\nu_2})$, where ξ^{ν_1} and η^{ν_2} are the highest powers of ξ and η appearing in $p(\xi, \eta)$; moreover, for given g in $\mathfrak{Q}(A_2^{\nu_2})$, one can by appropriately grouping terms on the right in (1) give $p(A_1, A_2)(f \otimes g)$ a meaning provided $f \in \mathfrak{Q}(A_1^{\mu_1})$, where μ_1 is the largest integer k for which $\sum_{l=0}^{\nu_2} a_{kl} A_2^l g$ is not zero; and similarly, for given f in $\mathfrak{Q}(A_1^{\nu_1})$ the requirement on g may be lightened. Let A' be the linear extension of the operator in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ defined by the equation (1) with these modifications. Then, if $\mathfrak{Q}(A_k^{\nu_k})$, $\mathfrak{Q}(A_k^{*\nu_k})$ are dense in $\mathfrak{H}_k$, $k=1, 2$, the transformation A' has a single-valued closure $p[A_1, A_2]$ with domain dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $(p[A_1, A_2])^* \supset \bar{p}[A_1^*, A_2^*]$. In particular, if A_1 and A_2 are symmetric and all a_{kl} real, then

$$(2) \qquad p[A_1, A_2] \subset p[A_1^*, A_2^*] \subset (p[A_1, A_2])^*.$$

Moreover, for the case chiefly under consideration, in which $A_2^* = A_2$, the author shows that

$$(3) \qquad p[A_1^*, A_2] = (p[A_1, A_2])^*$$

holds. This result is obtained through a preliminary characterization of the deficiency-spaces of $p[A_1, A_2]$; the projection operator F_λ of the manifold $\mathfrak{M}_\lambda$ of solutions of the equation $p[A_1^*, A_2]f - \bar{\lambda} f = 0$, $\Im(\lambda) \neq 0$, has the representation

$$(4) \qquad F_\lambda = \int_{-\infty}^{\infty} F_{1,\eta,\lambda} \otimes dE_2(\eta);$$

here $E_2(\eta)$ is the resolution of unity of A_2 and $F_{1,\eta,\lambda}$ is the projection operator of the manifold of zeros of $(A_{1,\eta,\lambda})^*$, where

$$A_{1,\eta,\lambda} = \sum_{k=0}^{\nu_1} \left(\sum_{l=0}^{n-k} a_{kl} \eta^l \right) A_1^k - \lambda.$$

The extension theory, in which formula (4) also plays a fundamental rôle, cannot be briefly described; it must suffice to say that the author achieves a representation for the isometric transformations U from $\mathfrak{M}_{-i}$ to $\mathfrak{M}_i$ which exhibits clearly their dependence on the operator $(A_{1,\eta,\pm i})^*$, $E_2(\eta)$. The self-adjoint extensions of $p[A_1, A_2]$ are of course determined by the operators U in accordance with the von Neumann theory.

The paper concludes with an examination of the case that $\mathfrak{H}_1 = \mathfrak{L}_2(a, b)$, (a, b) finite, $\mathfrak{H}_2 = \mathfrak{L}_2(c, d)$, with A_1^* defined in the customary way by a differential operator coincident with its formal adjoint. For this case, the preceding results permit the characterization of the self-adjoint extensions of $p[A_1, A_2]$ by means of boundary conditions whose formal statement is too complicated for description here. The application of this result to the case that A_2 is an integral operator is indicated briefly, and its application to the case $A_1 = -id/dx$, $A_2 = -id/dy$, $c = -\infty$, $d = \infty$, in somewhat more detail. Thus there is developed a representation for the self-adjoint boundary conditions associated with the differential operator

$$\sum_{k+l \leq n} a_{kl} \frac{\partial^k}{\partial x^k} \frac{\partial^l}{\partial y^l}$$

on the infinite strip $a < x < b$, $-\infty < y < \infty$.

J. W. Calkin (Chicago, Ill.).

Murray, F. J. The analysis of linear transformations. Bull. Amer. Math. Soc. 48, 76–93 (1942). [MF 6184]

A discussion of some aspects of the general problem of determining the structure of a linear transformation T mapping one Banach space B_1 onto another B_2. The author lists various specific problems, a solution of which seems in his opinion desirable to illuminate the general situation. He views the structure problem in the following form: Find projections E and F such that (1) TE"$=$"FT. The quotation marks suggest that the equality needs special interpretation. The possible unboundedness or absence of closure of the transformations considered are of course sources of great complication in the entire discussion. The author points out that a preliminary study of Banach spaces has to be made to determine the nature of the projections that exist in them, and whether there exist bases, discrete or continuous (resolutions of the identity). Equation (1) is now considered in the form (2) $TE(\lambda)$"$=$"$F(\lambda)T$, where $E(\lambda)$ and $F(\lambda)$ are resolutions of the identity of several types.

Although this discussion seems to include as a special case that in which T transforms B_1 into itself, it should be pointed out that here equation (1) is very secondary to the equation of reducibility (3) $TE = ET$; the program does not seem capable of assimilating this situation. Many examples of possible phenomena are to be found in the appendices.

E. R. Lorch (New York, N. Y.).

Pospíšil, Bedřich. Eine Bemerkung über stetige Verteilungen. Časopis Pěst. Mat. Fys. 70, 68–72 (1941). (German. Czech summary) [MF 5429]

The present note completes earlier researches of the author [Math. Ann. 117, 327–355 (1940); cf. these Rev. 2, 131 and "Von den Verteilungen auf Booleschen Ringen," still unpublished] into the partitions of a general Boolean algebra (ring) A. The author correlates the bicompact Hausdorff space e associated with A, the quotient-algebra K obtained from A by ignoring sets of first category in e, and the space (Banach lattice) H of all functions continuous on e. For example, the functions defined on the partitions of A are the functions continuous on e and measurable in K; H is the completed hull of the space of bounded continuous partitions of A; either the metric or the algebra of H defines A to within isomorphism. The ideas involved

relate to the representation theory of vector lattices [cf. M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. **27**, 83–87 (1941); cf. these Rev. **2**, 318 and the following review].

G. Birkhoff (Cambridge, Mass.).

Yosida, Kôsaku. On vector lattice with a unit. Proc. Imp. Acad. Tokyo **17**, 121–124 (1941). [MF 5391]

Let E be any Archimedean vector lattice with a strong unit e. The author proves that (i) modulo any maximal ideal (normal subspace) N, E is mapped homomorphically onto the real number system; (ii) the set of all such mappings represents E isomorphically as a subalgebra of the vector lattice $B(\mathfrak{R})$ of all bounded functions on the set $\mathfrak{R}$ of all maximal ideals of E; (iii) if neighborhoods of each maximal ideal N_0 are defined by finite sets of inequalities $|f_i(N)-f_i(N_0)|<\epsilon$ $(f_i \in E)$, then $\mathfrak{R}$ is a bicompact topological space. By construction of this topology, the elements of E are continuous functions on $\mathfrak{R}$; further, they are dense in the (Banach) lattice $C(\mathfrak{R})$ of all continuous functions on $\mathfrak{R}$. Finally, if E is (metrically) complete, then $E \cong C(\mathfrak{R})$; this is the Kakutani-Krein representation theorem.

G. Birkhoff (Cambridge, Mass.).

Nakano, Hidegorô. Teilweise geordnete Algebra. Jap. J. Math. **17**, 425–511 (1941). [MF 5642]

The author gives detailed proofs of results already announced [Proc. Imp. Acad. Tokyo **16**, 437–441 (1940); cf. these Rev. **2**, 343], with many amplifications. There are two appendices, one of which relates to a recent paper of Yosida [see the preceding review]. The exposition is self-contained. *G. Birkhoff* (Cambridge, Mass.).

Nakano, Hidegorô. Eine Spektraltheorie. Proc. Phys.-Math. Soc. Japan (3) **23**, 485–511 (1941). [MF 5407]

Using the results of his earlier paper [see the preceding review] and following ideas of Freudenthal [Nederl. Akad. Wetensch., Proc. **39**, 641–651 (1936)], the author gives a generalized theory for elements of a vector lattice. The relation of this to the usual spectral theory for self-adjoint operators on Hilbert space is easily seen by one familiar with the latter. The maximal ideals in the Boolean algebra of all "projectors" are treated as points. At each such point, given elements a and b of a suitably restricted vector lattice, just one characteristic value for b relative to a as unit exists. This may be finite or infinite. The resulting function on the set of all maximal ideals describes the spectrum of b relative to a. *G. Birkhoff* (Cambridge, Mass.).

Plessner, A. I. Spectral theory of linear operators. I. Uspekhi Matem. Nauk **9**, 3–125 (1941). (Russian) [MF 5078]

The present survey is based on a course of lectures delivered by the author in 1938–1939 at the Moscow University. It contains, together with an introduction and fundamental notions of the theory of Hilbert spaces and linear operators in Hilbert spaces, the foundations of the spectral theory of linear operators (bounded and not bounded) and of extensions of self-adjoint operators. The spectral decomposition is treated by a method analogous to the complex methods of Poincaré-Hellinger, but presenting some new details. The exposition is compact and elegant. The theory of functions of operators is reserved for the second part of the survey. *J. D. Tamarkin.*

***Hellinger, E. D. Spectra of quadratic forms in infinitely many variables.** Northwestern University Studies in Mathematics and the Physical Sciences, no. 1: Mathematical Monographs, vol. 1, pp. 133–172. Graduate School, Northwestern University, Evanston, Ill., 1941. $2.25.

This memoir presents an effective description of the spectral theory by means of examples and without proofs. The finite and completely continuous cases are discussed first. Jacobi forms are used to illustrate the continuous spectra of bounded forms, double spectra and the expression of the spectral theory by means of Hellinger integrals. The results of the orthogonal equivalence theory for real forms are stated and a method for determining the basis functions used in the Hellinger spectral integrals is given. In the last chapter three examples of unbounded forms are presented. These correspond respectively to a self-adjoint operator, a non-maximal symmetric operator and a maximal symmetric operator. As an introduction to the spectral theory and for supplementary reading, this is a notable addition to the literature. *F. J. Murray* (New York, N. Y.).

Ambrose, Warren and Kakutani, Shizuo. Structure and continuity of measurable flows. Duke Math. J. **9**, 25–42 (1942). [MF 6339]

Let Ω be a space with a measure defined on it (a measure space). A measurable flow on Ω is a family of measure preserving transformations T_tP such that $T_sT_t=T_{s+t}$, and that T_tP is (t, P)-measurable. Two such flows, on spaces Ω_1, Ω_2, are called isomorphic if (after removing invariant sets of measure 0 from the spaces) some 1-1 measure preserving transformation takes Ω_1 into Ω_2 in such a way that the flow transformations go into each other. A flow under a function is defined as follows. Let $f(\omega)$ be a positive measurable and integrable function defined on a measure space Ω_0. Let S be a 1-1 measure preserving transformation taking Ω_0 into itself, such that $\sum_1^\infty f(S^n\omega)=\sum_1^\infty f(S^{-n}\omega)=+\infty$. Let Ω be the space of pairs (ω, u) with $\omega \in \Omega_0$ and u real, $0 \leqq u \leqq f(\omega)$. Measure on Ω is defined in the usual multiplicative way in terms of Ω_0-measure and Lebesgue u-measure. The transformation $T_t(\omega, u)$ takes (ω, u) into $(\omega, u+t)$ as t increases from 0, until $u+t=f(\omega)$, then the image point jumps to $(S\omega, 0)$ and continues up the u-line over $S\omega$ as before, etc. The authors prove that, neglecting a pathological case, any measurable flow is isomorphic to a flow built under a function, or its space is the sum of denumerably many invariant subspaces on which the flow is isomorphic to such flows. This was proved in the metrically transitive case by Ambrose [Ann. of Math. (2) **42**, 723–739 (1941); these Rev. **3**, 52]. If conditions are imposed on Ω relating to the approximation of the field of measurable sets by means of denumerable subfields, it is shown that a given flow on Ω is isomorphic to a flow $\{T_t'(P')\}$ on a separable metric space Ω'. Moreover, $T_t'(P')$ is continuous in (t, P'); the open sets of Ω' are measurable and have positive measure; the measure of a measurable Ω'-set is the G.L.B. of the measures of containing open sets. The hypotheses on Ω'-measure are satisfied for example by Lebesgue measure.

J. L. Doob (Princeton, N. J.).

Ambrose, Warren, Halmos, Paul R. and Kakutani, Shizuo. The decomposition of measures. II. Duke Math. J. **9**, 43–47 (1942). [MF 6340]

The authors prove that a measurable flow on a measure space Ω [cf. the preceding review] can be decomposed into

ergodic parts. This theorem was proved by von Neumann [Ann. of Math. (2) **33**, 587–642 (1932), in particular, pp. 601–618] who imposed topological restrictions on Ω. The authors' restrictions are non-topological in nature. Using the fact [cf. the preceding review] that the given flow can be assumed to be one built under a function the problem is reduced to the analogous problem for the iterates of a single measure preserving transformation. The latter problem was treated by Halmos [Duke Math. J. **8**, 386–392 (1941); these Rev. **3**, 50] whose treatment must, however, be extended to be applicable. *J. L. Doob.*

Halmos, Paul R. Square roots of measure preserving transformations. Amer. J. Math. **64**, 153–166 (1942). [MF 5995]

Let X be a metric, complete, separable space in which an l-measure in the sense of von Neumann [Ann. of Math. (2) **33**, 574–586 (1932)] is defined such that $mX<\infty$. Let T be a one-to-one measure preserving transformation of X into itself. The transformation T has a square root if there exists a transformation S of the same type such that $S^2=T$ (almost everywhere). The problem considered in this paper is the interesting one of determining necessary and sufficient conditions that T have a square root. The importance of the characteristic value -1 in this consideration is shown by a proof that if T has a square root, then -1 is a characteristic value of T of even multiplicity. It immediately follows that if -1 is a characteristic value of a metrically transitive T then T does not have a square root. On the other hand, if T is metrically transitive and has a pure point spectrum not containing -1, then T has a square root. The more difficult result is then obtained that in the latter theorem the hypothesis of metrical transitivity is unnecessary. Finally, under the hypothesis that T has a pure point spectrum and there exists a nowhere vanishing real valued characteristic function belonging to the characteristic value -1, there is obtained a necessary and sufficient condition that T have a square root, this condition involving the existence of an invariant function and of a set of characteristic functions with certain properties. Examples illustrating various possibilities in this connection are displayed. *G. A. Hedlund* (Charlottesville, Va.).

Oxtoby, J. C. and Ulam, S. M. Measure-preserving homeomorphisms and metrical transitivity. Ann. of Math. (2) **42**, 874–920 (1941). [MF 5521]

This paper undertakes a thorough and detailed study of the group of measure preserving and measurability preserving automorphisms (homeomorphisms into itself) in polyhedra, their metrical transitivity, equivalence, and the whole bearing of such questions on ergodic theory. An idea of its contents may be given by citing the following theorems which are among the most typical and interesting in the paper: Theorem 1. Let E be any regularly connected polyhedron (that is, the sort of manifold always occurring in dynamics) of dimension $r\geqq 2$, and let μ be any r-dimensional Lebesgue-Stieltjes measure in E. In the space $M[E,\mu]$ of measure-preserving automorphisms of E (distance in M being defined in a simple and natural way) the metrically transitive automorphisms form a residual G_δ set. (A G_δ set is a countable intersection of open sets; a residual set is the complement of a set of the first category, that is, of a sum of countably many nowhere dense sets.) Thus the metrically transitive automorphisms not only exist: they are indeed the "rule" in $M[E,\mu]$. Theorem 2. Under the above conditions, if $r\geqq 3$, there exists a continuous flow of E which is metrically transitive with respect to μ and which leaves all singular points of E fixed. Theorem 10. Let h be an automorphism of the r-dimensional unit cube R, $r\geqq 2$. In order that there exist an automorphism g such that ghg^{-1} preserves Lebesgue measure it is necessary and sufficient that in every sphere there exist a perfect set to which some point returns with positive frequency under iteration of h, and that these perfect sets can all be chosen outside of an arbitrarily prescribed countable set. (A point p of C is said to return with positive frequency if $\lim_{n\to\infty} n^{-1}\sum_{r=1}^{n} f_C(T^r p)>0$, where f_C is the characteristic function of C.)

B. O. Koopman (New York, N. Y.).

Carathéodory, C. Bemerkungen zum Riesz-Fischerschen Satz und zur Ergodentheorie. Abh. Math. Sem. Hansischen Univ. **14**, 351–389 (1941). [MF 5451]

The author develops a theory of measure and integration on a countably additive Boolean algebra (=soma) [cf. C. Carathéodory, S.-B. Bayer. Akad. Wiss. **1938**, 27–68]. The relations between almost everywhere convergence and mean convergence are discussed on soma. Two examples are given as illustrations: Riesz-Fischer's theorem and ergodic theorems (G. D. Birkhoff's individual ergodic theorem and von Neumann's mean ergodic theorem). Wiener's treatment of these ergodic theorems is also discussed. In these discussions the following inequality of Birkhoff plays a fundamental rôle: Let T be a measure preserving transformation of a measure space S onto itself, and let $f(x)\varepsilon L_1$. Then $\alpha\cdot m(E^*(\alpha))\leqq\int_{E^*(\alpha)} f(x)dx$, where

$$E^*(\alpha)=E_x[\sup_n\{f(x)+f(T(x))+\cdots+f(T^{n-1}(x))\}/n>\alpha]$$

and α is any real number. *S. Kakutani.*

RELATIVITY

Levi-Civita, Tullio. Il punto materiale in meccanica relativistica. Pont. Acad. Sci. Acta **5**, 53–54 (1941). [MF 5886]

Henderson, Archibald. A new geometrical interpretation of Einstein's special relativity theory. J. Elisha Mitchell Sci. Soc. **57**, 284–293 (1941). (1 plate) [MF 6178]

The author states two postulates for the theory of special relativity which are not independent. He then gives a geometrical interpretation of a Lorentz transformation connecting two observers with coordinates (X, T) and (X', T') in terms of a complicated geometrical figure in the T, T' plane. *A. H. Taub* (Princeton, N. J.).

Heckmann, O. Geodätische Linien und Newtonsche Bewegungsgleichungen. Abh. Math. Sem. Hansischen Univ. **14**, 192–196 (1941). [MF 5444]

The object of this paper is to provide an answer to the question: Under what conditions are the equations of motion of Newtonian mechanics formally identical with the equations of motion (geodesics) of general relativity? The author states that this question was suggested by the study

of papers by McCrea and E. A. Milne [Quart. J. Math., Oxford Ser. 5, 73–80 (1934)]. The Newtonian equations of motion are taken in the form

$$\frac{d^2x^i}{dt^2}+\frac{\partial\Phi}{\partial x^i}=0, \qquad i=1, 2, 3, \tag{1}$$

where $\Phi=\Phi(x^1, x^2, x^3, t)$ is a potential function. Upon writing the equations for the geodesics associated with the Riemann metric $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ ($\mu, \nu=1, 2, 3, 4$) in terms of the parameter $x^4=t$ [see Eisenhart, Riemannian Geometry, Princeton University Press, 1926, p. 50, eq. (17.7)] and comparing with equations (1), the author obtains a set of conditions to be satisfied by the Christoffel symbols $\Gamma^\sigma_{\beta\gamma}$. These results, used in conjunction with the integrability conditions

$$g_{\sigma\alpha}R^{\cdots\sigma}_{\gamma\beta\delta}+g_{\sigma\beta}R^{\cdots\sigma}_{\gamma\delta\alpha}=0$$

(where $R^{\cdots\sigma}_{\alpha\beta\gamma}$ is the Riemann-Christoffel tensor), yield a set of partial differential equations which must be satisfied by the $\Gamma^\sigma_{\beta\gamma}$. A discussion of these equations leads to the determination of the fundamental tensor $g_{\mu\nu}$. From this result it is concluded that the only four dimensional metric for which the geodesics have equations which are exactly of the form of the Newtonian equations of motion is one of constant curvature. *M. A. Basoco* (Lincoln, Nebr.).

Clark, G. L. The derivation of mechanics from the law of gravitation in relativity theory. Proc. Roy. Soc. London. Ser. A. **177**, 227–250 (1941). [MF 3938]

It was shown by Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] that the relativity field equations of gravitation in free space imply restrictions on the motions of singularities in the field defined by these equations, and that these restrictions lead, in suitable approximation, to the Newtonian equations of motion. In higher approximation they lead to equations of motion that were integrated by H. P. Robertson [Ann. of Math. (2) **39**, 101–104 (1938)] and by Clark and Eddington [Proc. Roy. Soc. London. Ser. A. **166**, 465–475 (1938)]. These results were accomplished by the introduction of certain integral conditions obtained by considering two-dimensional surfaces enclosing singularities of the field.

In the present paper, these integral conditions are not used. Instead, it is assumed that the gravitational equations in free space do imply restrictions on the motion, and the explicit equations of motion in several dynamical problems are calculated to various orders of approximation by a method that is essentially as follows. Starting from a solution of the field equations assumed as known, the author considers the effect of small additive terms representing the motion of small masses at low velocities. It is assumed that these additive terms can be represented by fairly general forms containing a number of undetermined constants. After substitution in the field equations, it is found that certain terms must be equated to zero. In each instance, it turns out that these terms have a form that suggests the equations of motion, and they are so interpreted.

The most interesting parts of the paper deal with the application of this method to cases where explicit solutions have not been obtained. (1) The case of a non-spherical body, moving in a known static field, for which the "angular equations of motion" are obtained in approximation. (2) The case of a number of isolated particles rotating about their center of gravity. This leads to the well-known equations of Euler. (3) The case of a rotating rod. The author here studies the loss of gravitational energy. Since the methods used in previous studies of this problem have been subjected to various criticisms, it is of interest that the present method leads to the same results as those obtained by Eddington [Mathematical Theory of Relativity, 2nd ed., Cambridge University Press, 1924]. As a final application, he studies the internal problem of a rotating mass of liquid of constant density. The result obtained is that "a possible form of the free surface is a spheroid."

Due to the nature of the method, it is not possible for it to prove the fundamental theorem that the equations of motion are implied by the field equations. What has been demonstrated here is that if there exists a solution of the field equations of the particular form assumed by the author, then it follows that there must exist restricting conditions which may be identified with equations of motion. The importance of the results therefore lies above all in the form of these restricting conditions. The method of approximation is not formalized by the use of power series, which fact makes certain parts of the paper difficult to read. The word acceleration occurs in place of the word velocity in the next to the last sentence of Section 3.

H. C. Levinson (Princeton, N. J.).

Narlikar, V. V. The gravitational equations of motion in relativity. Proc. Indian Acad. Sci., Sect. A. **14**, 187–195 (1941). [MF 5871]

The author investigates whether the equations of motion as formulated by Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] are consistent with the equations of the geodetic line. He shows this consistency when one of the two masses is very small. This, however, does not settle the more general case of two arbitrary masses, since the most important expression characterizing the interaction is neglected by the author due to his assumption that one of the masses is small. *L. Infeld* (Toronto, Ont.).

Fock, V. A. Sur les intégrales du centre de gravité dans le problème relativiste de deux masses finies. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 25–27 (1941). [MF 5849]

This paper deals with the motion of the center of gravity in a two-body problem treated according to general relativity theory. The same result obtained by H. P. Robertson [Ann. of Math. (2) **39**, 101–104 (1938)] and based upon the equations of Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] is obtained more explicitly. The result is: there is no acceleration of the center of gravity.

L. Infeld (Toronto, Ont.).

Wallace, P. R. Relativistic equations of motion in electromagnetic theory. Amer. J. Math. **63**, 729–740 (1941). [MF 5621]

The author considers a system of equations consisting of the equations of general relativity theory and Maxwell's equations. He shows that these field equations determine the motion of charged singularities. This has previously been shown in the case of a purely gravitational field by Einstein-Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)]. The present paper gives a generalization of these results. The approximation method used is based essentially on the assumption that the velocities of the particles are small compared with the velocity of light. Besides the already known purely gravitational terms, the equations of motion also contain the purely electromagnetic and mixed gravitational-electromagnetic terms. The equa-

tions of motion are investigated both in the case of a "standing wave" and "retarded" field. The two body problem is solved for the case of two oppositely charged particles. It is shown that stable elliptical orbits are possible only for a standing wave solution, that is, when the radiation terms are omitted. *L. Infeld* (Toronto, Ont.).

Cattermole, J. Relativity theory of electromagnetism. Philos. Mag. (7) **33**, 215–225 (1942). [MF 6448]

The equations of the geodesics in a "Kaluzean" space, that is, a five dimensional space with a line element

$$\gamma_{\alpha\beta}=g_{\alpha\beta}+(K_\alpha K_\beta/K_5^2)\gamma_{55}, \quad \alpha,\beta=1,\cdots,5,$$

when $g_{a5}=0$ and g_{ab} for $a, b=1, 2, 3, 4$ is the metric of the four dimensional space time of general relativity, K_a is the vector potential and K_5 is a constant, are solved by neglecting gravitational effects (assuming $g_{ab}=\delta_{ab}$) and assuming various forms of K_a. Since the Kaluza five dimensional theory is another mathematical formulation of the theory of relativity it is evident that under the assumptions made the equations of the geodesics reduce to those of special relativity for a charged particle in an electromagnetic field and the solutions obtained are just those that special relativity gives. *A. H. Taub* (Princeton, N. J.).

Gentile, Giovanni. Sulle equazioni d'onda relativistiche di Dirac per particelle con momento intrinseco qualsiasi. Nuovo Cimento (N.S.) **17**, 5–12 (1940). [MF 5607]

The relativistic wave equations for a particle of arbitrary spin obtained by Dirac [Proc. Roy. Soc. London. Ser. A. **155**, 447–459 (1936)] are written in two component spinor form. The author uses a method which differs from that given by Dirac in the paper quoted. *A. H. Taub.*

de Beauregard, Olivier Costa. Le tenseur antisymétrique densité de moment pondéromoteur propre. C. R. Acad. Sci. Paris **211**, 499–501 (1940). [MF 5359]

Weiss, P. On some applications of quaternions to restricted relativity and classical radiation theory. Proc. Roy. Irish Acad. Sect. A. **46**, 129–168 (1941). [MF 5660]

The author uses the (1-1) correspondence between sets of four complex numbers X^i and biquaternions, quaternions over the complex field, to replace tensor equations of special relativity by quaternion equations. The author does not use the binary spinor formalism and as a result his calculations are carried out in a special frame of reference and proofs of the invariance of results are included. The method is first applied to the problem of expressing the ordinary coordinates of special relativity in terms of two suitably chosen angular variables, the proper time and the retarded distance. The retarded distance from an observer to a point on the light cone whose vertex is at the observer is defined to be the radius of a sphere centered at the observer on which the point appears to be. These results are then used to obtain a quaternion expression for the retarded electromagnetic field of a moving point charge and a derivation of the classical radiation formula. The latter is formulated in terms of a variation principle. *A. H. Taub.*

Michels, Walter C. and Patterson, A. L. Special relativity in refracting media. Phys. Rev. (2) **60**, 589–592 (1941). [MF 5473]

It is assumed that: (1) the space-time coordinates of two observers O and O' in free space, where O' moves with a constant velocity with respect to O, are related by a Lorentz transformation; (2) the space-time coordinates of two similar observers $\bar{O}$ and $\bar{O}'$ moving in a refractive (but not dispersive) medium are related by a Lorentz transformation with c replaced by $\bar{c}=c/n$ (n is the refractive index of the medium); (3) the space-time coordinates of an observer O_0 outside the medium and at rest with respect to it are related by the identity transformation to those of an observer $\bar{O}_0$ inside the medium and at rest with respect to it. The velocity measurements of an observer inside the medium are expressed in terms of an outside observer's by expressing each in terms of the observers $\bar{O}_0$ and O_0, respectively, and using assumption (3). It is then shown that an observer outside the medium and moving with a velocity $-U_0$ with respect to the medium assigns the velocity

$$v=(c/n-U_0)/(1-U_0/nc)$$

to light traveling inside the medium, which is in agreement with the formula derived in the usual treatment of special relativity. The authors then consider photons traveling from free space into the medium and find that, if the energy of the photon is the same inside and outside the medium, the momentum changes in accordance with

$$\bar{p}/p=n=\sin I/\sin R,$$

where $\bar{p}$ and p are the momenta inside and outside the medium, respectively, and I and R are the angles of incidence and refraction, respectively. *A. H. Taub.*

MECHANICS

***Synge, J. L. and Griffith, B. A. Principles of Mechanics.** McGraw-Hill Book Company, Inc., New York, 1942. xii+514 pp. $4.50.

The reproduction of the rather conservative chapter headings would not give a clear idea of the broad scope of this excellent book. With the exception of a chapter on the special theory of relativity and a section on electron optics, the ground covered is almost the same as that covered in the classical French textbooks of "Mécanique rationnelle," the careful, clear and concise exposition of which has evidently been taken as a model, the traditional dullness of the examples of which has, however, been avoided by the introduction of subjects from engineering and physics. Plane mechanics and space mechanics are dealt with in separate parts of the book which, as far as possible, have been made self-consistent. Each chapter is followed by a summary. The vector notation, though introduced in the first chapter, is used sparingly in the part on plane mechanics and only so far as sums or differences of vectors are involved. The first chapter of the second part deals with the scalar and vector products. The dyadic products are not introduced. Consequently, the vector notation seems to become inadequate in the chapter concerned with kinetics of rigid bodies. In many places the definitions given differ slightly from those usually adopted. The only case in which difficulties may arise from such a modification seems to be the definition of a "just rigid frame" which still covers kinematically critical forms. *W. Prager.*

Siegel, Carl Ludwig. On the integrals of canonical systems. Ann. of Math. (2) **42**, 806–822 (1941). [MF 4973]

Consider a canonical system of differential equations

$$\dot{x}_k = H_{y_k}, \quad \dot{y}_k = -H_{x_k}, \qquad k=1, \cdots, n, \tag{1}$$

where the real function H is independent of t and is an analytic function of the $2n$ variables $x_1, \cdots, y_n$ in a neighborhood of the origin. Assume that the number n of degrees of freedom is at least 2, and that H, H_{x_k}, H_{y_k} all vanish at the origin. Denoting the $2n$ variables by $z_1, \cdots, z_{2n}$, the system (1) may be written

$$\dot{z}_k = \sum_{l=1}^{2n} a_{kl} z_l + R_k, \qquad k=1, \cdots, 2n,$$

where R_k is a power series in $z_1, \cdots, z_{2n}$ beginning with terms of order greater than the first. Let $\lambda_1, \cdots, \lambda_{2n}$ be the characteristic roots of the matrix (a_{kl}); they may be arranged so that $\lambda_{k+n} = -\lambda_k$ $(k=1, \cdots, n)$. It is assumed that $\lambda_1, \cdots, \lambda_n$ are linearly independent with respect to the field of rational numbers. By the appropriate contact transformation, the function H is expressed in terms of variables ξ_k, η_k $(k=1, \cdots, n)$ in the form

$$H = \sum_{k=1}^{n} \lambda_k \zeta_k + R, \qquad \zeta_k = \xi_k \eta_k, \tag{7}$$

where R is a power series in the n variables $\zeta_1, \cdots, \zeta_n$ beginning with terms of degree higher than the first. This leads to a solution of (1).

If we consider the system (1) when H is a function of t, difficulties arise with regard to the contact transformation. If we define sum, difference, product, quotient and derivative of power series in a formal algebraic manner, we can perform these operations with divergent power series, and make formally an appropriate contact transformation. Assume that the power series H converges in a neighborhood of the origin, and that $\lambda_1, \cdots, \lambda_n$ are pure imaginary. Arrange the coefficients of H in a certain order $h_1, h_2, h_3, \cdots$. The corresponding systems $(h_1, h_2, h_3, \cdots)$ form the points of a space Σ. A point of Σ is called singular if the transformation of H into the normal form (7) cannot be performed by a convergent contact transformation; otherwise it is a regular point. The author gives the following: Theorem 1. Let $(c_1, c_2, c_3, \cdots)$ be a point of Σ and $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ an arbitrary sequence of positive numbers. Then a singular point $(h_1, h_2, h_3, \cdots)$ of Σ exists in the domain $c_k - \epsilon_k < h_k < c_k + \epsilon_k$ $(k=1, 2, 3, \cdots)$. Theorem 2. Under the hypotheses of Theorem 1 there exists a point $(h_1, h_2, h_3, \cdots)$ in the domain $c_k - \epsilon_k < h_k < c_k + \epsilon_k$ $(k=1, 2, 3, \cdots)$ such that any convergent integral of the corresponding canonical system (1) is a power series in the single variable H. Another theorem of a similar character is proved.

E. J. Moulton (Evanston, Ill.).

Kasner, Edward and Mittleman, Don. Extended theorems in dynamics. Science (N.S.) **95**, 249–250 (1942). [MF 6334]

Suppose given two real analytic functions $f(x, y)$ and $g(x, y)$ defined in the vicinity of the origin. Let C_k (for integral values of k) be defined as the curve whose parametric equations $x=x(t)$, $y=y(t)$ satisfy the conditions:

$$\frac{d^i x(0)}{dt^i} = \frac{d^i y(0)}{dt^i} = 0, \quad i=0, 1, 2, \cdots, k-1.$$

$$\frac{d^k x}{dt^k} = f(x, y), \quad \frac{d^k y}{dt^k} = g(x, y).$$

Supposing that $f(0, 0)$ and $g(0, 0)$ are not both zero, there are various results concerning the nature of contact at $(0, 0)$ between C_1 and C_r. The prototype of such results, discussed in the present note, was given by one of the authors many years ago for the simple case $r=2$ to the effect that C_2 is tangent to C_1 and has one third its curvature.

D. C. Lewis (Durham, N. H.).

Caldirola, P. Su alcune relazioni fra le proprietà geometriche di una V_n e la dinamica delle particelle. Nuovo Cimento (N.S.) **17**, 69–73 (1940). [MF 5608]

The first part of the paper deals with a congruence of geodesics in a V_n and the possibility of regarding them as projections of null geodesics in a V_{n+1}. The argument makes use of the Hamilton-Jacobi partial differential equation, but is not entirely clear, since the author fails to distinguish between normal and skew congruences; the relation $\lambda_i = \partial f / \partial x^i$ holds only for a normal congruence. A description is given of hypotheses made by various authors in order to include the charge and spin of a particle in relativistic dynamics by increasing the dimensionality of space-time to five or six.

J. L. Synge (Toronto, Ont.).

MacColl, L. A. Trajectories of monoenergetic electrons, in an arbitrary static electromagnetic field, in the neighborhood of a given trajectory. J. Math. Phys. Mass. Inst. Tech. **20**, 355–369 (1941). [MF 6106]

This paper deals with the focal properties of trajectories. The plan is to study the behavior of a trajectory T adjacent to a basic trajectory T_0 by resolving the normal displacement from T_0 to T along the principal normal and binormal of T_0. If the components of this displacement are ξ, η, then these small variables satisfy linear homogeneous differential equations of the second order, the independent variable being σ, the arc length of T_0. The classification of focal properties turns on consideration of the rank r of the 4×4 matrix formed by the values of four independent solutions for ξ, η, evaluated for $\sigma=\sigma_1$ and $\sigma=\sigma_2$. If Q_1 is the normal plane of T_0 at σ_1 and Q_2 the normal plane at σ_2, then the trajectories through an arbitrary point of Q_1 behave as follows: (i) they spread over Q_2 if $r=4$; (ii) they give a line image on Q_2 if $r=3$; (iii) they give a point image on Q_2 if $r=2$. Although the trajectories satisfy a variational principle, little explicit use is made of this fact. [For W. R. Hamilton's general treatment of focal properties, see Mathematical Papers, Cambridge University Press, Cambridge, England, 1931, vol. 1, pp. 270–277.] *J. L. Synge.*

Buchanan, Daniel. Second genus crossed orbits. Canadian J. Research. Sect. A. **20**, 11–24 (1942). [MF 6236]

In previous papers [Proc. Roy. Soc. London. Ser. A. **114**, 490–516 (1927); Rend. Circ. Mat. Palermo **60**, 1–21 (1931)] the author obtained first genus periodic orbits for the electrons in the crossed orbit model of the normal helium atom. The problem considered in the present paper is to determine second genus periodic orbits. He defines his "first genus" and "second genus" orbits as follows: Given $dx_i/dt = X_i(x_j, \epsilon, t)$, X_i being analytic in x_j and ϵ and not containing t explicitly. Let $x_i = \theta_i(\epsilon, t)$ be a set of periodic solutions of period 2π. These are the first genus solutions. Now let $\epsilon = \epsilon_0(1+\lambda)$ and $x_i = \theta_i(\epsilon_0, t) + y_i$. If we can determine $y_i = \psi_i(\epsilon_0, \lambda, t)$ having the period $2\pi N$ (1+a power series in λ), N an integer, then

$$x_i = \theta_i(\epsilon_0, t) + \psi_i(\epsilon_0, \lambda, t)$$

is of the second genus. The differential equations for the motion of the electrons are set up. The scale factor ϵ and $t-t_0=A(1+\delta)^{\frac{1}{2}}t$ are used to obtain the first genus solutions. Then ϵ is set equal to $\epsilon_0(1+\lambda)$ wherever it occurs explicitly, and $1+\delta$ is replaced by $(1+\delta)(1+\gamma)$. The differential equations for the second genus orbits are then integrated as power series in λ. Next he shows that γ can be determined as a power series in λ such that a proper determination of the coefficients of the series and the constant of integration will render the solutions periodic and will satisfy certain initial conditions. The convergence of the solutions is assured by a theorem of Macmillan, and they converge for all finite values of the time provided λ is sufficiently small.

H. E. Buchanan (New Orleans, La.).

Casadio, Giuseppina. Un teorema sulle masse variabili. Boll. Un. Mat. Ital. (2) **3**, 30–32 (1940). [MF 5573]

The rectilinear oscillatory motion of a body with adiabatically increasing mass is studied under the action of a force of elastic type. It is shown that the amplitude increases if we take as our law of motion: force = mass times acceleration, while it decreases if we assume that force = the rate of change of momentum. *D. C. Lewis* (Durham, N. H.).

Astronomy

***Wintner, Aurel. The Analytical Foundations of Celestial Mechanics.** Princeton Mathematical Series, v. 5. Princeton University Press, Princeton, N. J., 1941. xii+448 pp. $6.00.

The general scope and arrangement of the book may be inferred from the successive chapter headings and lengths: I. Dynamical operations, 55 pages. II. Local and non-local questions, 54 pages. III. Dynamical systems, 66 pages. IV. The problem of two bodies, 55 pages. V. The problem of several bodies, 114 pages. VI. Introduction to the restricted problem, 66 pages. Historical notes and references, 32 pages.

In contrast with earlier books on celestial mechanics, this one makes consistent use of recent developments in algebra, and differential equations which utilize the condensed notations of n-vectors, matrices, and so on. This abbreviates the presentation markedly. [The condensation is perhaps carried too far at times, leading to sentences which seem unnecessarily intricate.] The first third of the book, approximately, has been included in lectures by the author for graduate students in physics and mathematics as "an introduction into the pure analysis of theoretical dynamics and of the theory of perturbations." Chapter V attempts to give an account of our present knowledge of the problem of three or more bodies. Chapter VI includes sections on the lunar theory dealing with the fundamental mathematical questions of the motion of the moon, stopping "at the border of the still unknown land of the 'small divisors' in classical celestial mechanics." *E. J. Moulton.*

Siegel, C. L. On the modern development of celestial mechanics. Amer. Math. Monthly **48**, 430–435 (1941). [MF 5535]

This is an expository paper, presenting briefly a statement of results obtained for the n body problem. Included are theorems by Bruns, by Poincaré, and by Sundman.

E. J. Moulton (Evanston, Ill.).

***Hagihara, Yusuke. Application of matrix method to integration problems in celestial mechanics.** Astronomical Papers dedicated to Elis Strömgren, pp. 58–81. Einar Munksgaard, Copenhagen, 1940.

Applications of H. F. Baker's method of solving linear differential equations [Philos. Trans. Roy. Soc. London. Ser. A. **216**, 129–186 (1916)]. Lindstedt's equation

$$\frac{d^2x}{dt^2}+(n^2+h+a\cos\mu nt)x=0,$$

n and μ being integers and h and a small numbers, and the equations that occur in Goldsbrough's theory of Saturn's rings are treated in full. An outline is presented of the treatment of the equations arising in the theory of planetary motion in commensurability regions. The equations treated are not those of the actual problems in celestial mechanics but curtailed equations in which infinite series are reduced to a few terms. The author states this limitation but stresses as the importance of the method the simplicity of the operations and its advantages in dealing with questions of convergence. *D. Brouwer* (New Haven, Conn.).

***Sundman, Karl F. Démonstration nouvelle du théorème de Poisson sur l'invariabilité des grands axes.** Astronomical Papers dedicated to Elis Strömgren, pp. 263–273. Einar Munksgaard, Copenhagen, 1940.

Poisson's theorem on the invariability of the major axes of the planetary orbits has been demonstrated in several ways. Sundman gives a new and elegant demonstration of the theorem. Using Tisserand's notation $\mu_0=m_0$, $\mu_1=m_0+m_1$, $\cdots$, $\mu_i=m_0+m_1+\cdots+m_i$, he develops the function

$$V=U-k^2\sum_{i=1}^{n}\frac{(m_0+m_i)\mu_{i-1}m_i}{\mu_i r_i}$$

into power series in m_i. The μ_i are considered as constant parameters. He succeeds in showing that the square root of the major axis is composed of three terms of the form $\sqrt{a_i}=L_i=L_{i0}+\Pi_1+\Pi_2 t+\Pi_2'$, where Π_1, Π_2 and Π_2' are periodic functions of t and Π_2, Π_2' are of the second order. Approximations of the third order are considered.

H. E. Buchanan (New Orleans, La.).

Sémirot, Pierre. Application de la transformation de Sundman au problème d'Euler. C. R. Acad. Sci. Paris **212**, 848–850 (1941). [MF 5034]

The author considers the motion of a particle which is attracted toward two fixed centers by forces varying according to the Newtonian law. By the aid of the transformation used by Sundman he obtains classical formulas for the integration of the appropriate differential equations.

E. J. Moulton (Evanston, Ill.).

Subbotin, M. F. Sur le calcul des inégalités séculaires. I. Solution nouvelle du problème de Gauss. Astr. J. Soviet Union [Astr. Žurnal] **18**, 35–50 (1941). (French. Russian summary) [MF 4272]

The calculation of the secular perturbations of the first order reduces, for its essential part, to the celebrated problem of Gauss: to determine the attraction due to an elliptic ring in which the density of any element is proportional to the area of the sector having the element as base and the vertex at one of the foci of the ring. Known methods of solution of the problem are quite general but do not take into account completely the particular circumstances which

obtain in the solar system. In the present memoir the author gives new solutions based essentially on the smallness of the eccentricity of the perturbing planet. In this part I of the memoir he develops the analytic formulas which he says lead to shorter calculations than those of other known methods. He states that part II contains applications to the secular perturbations of the small planets.

E. J. Moulton (Evanston, Ill.).

Buchanan, Daniel. Trojan satellites (Limiting case). Trans. Roy. Soc. Canada. Sect. III. (3) **35**, 9–25 (1941). [MF 6021]

The author considers periodic orbits near the equilateral triangle positions in the restricted problem of three bodies. The special case considered is that in which $\mu(1-\mu)=1/27$, where μ and $1-\mu$ are the masses of the two finite bodies. The method is identical with that set forth in Moulton's Periodic Orbits [Carnegie Inst. Publ. **161**, Carnegie Institution of Washington, Washington, D. C., 1920] and the solutions are given as power series in the scale factor ϵ. He adopts, following Moulton, the transformation $t=(1+S)t$ and determines δ as a power series in ϵ. It turns out that $72\delta=\pm\sqrt{1770}\epsilon+\cdots$, and hence there are orbits of two different periods. When the plus sign is used a long period results and when the minus is used there is a short period orbit. These results agree with those of Pedersen who used Fourier series instead of power series. *H. E. Buchanan.*

Prey, A. Über die periodischen Bahnen in der Nähe der Librationszentra L_4 und L_5. Astr. Nachr. **271**, 265–274 (1941). [MF 5790]

The complete solution of the problem, as given by Charlier, is discussed in detail. This is done in order to establish again what was given by him concerning the perturbations produced by Jupiter. Charlier established the existence of two periodic solutions, one with a very long period and another with a short period. These he called *d* and *e* orbits. The author finds that, in addition to these two known solutions, there is also a secular perturbation, proportional to the time, of the perihelion point. He points out that it would be advantageous not to proceed as is usually done, but to adopt as an intermediary path one of the *d*-class, long period orbits of Charlier. From this set-up he computes a table from which the elements of the orbits of the Trojan planets may be found. Finally, the author shows how the differential equations can be thrown into the classical form by introducing an additive term to the perturbative function, but he does not use this result. *H. E. Buchanan.*

***Chandrasekhar, S. The fundamental principles of stellar dynamics.** Astronomical Papers dedicated to Elis Strömgren, pp. 1–24. Einar Munksgaard, Copenhagen, 1940.

The paper gives a concentrated review of the problems encountered in the dynamics and kinematics of stellar systems. New ways of approach are not attempted. The classical method of ascribing to the system a velocity-distribution and a potential field is the subject of a critical examination. The purpose of the paper is to clarify some of the rather hazy points of the theory, like the definition of a "local standard of rest." A recalculation of the "time of relaxation" of the galactic system is given, errors in earlier calculations being corrected. The "equation of continuity" for the distribution function is discussed. It is suggested that the most promising line of attack is to investigate under what conditions this equation of continuity allows of a solution of the form of a generalized Schwarzschild distribution function. Finally the stability of the orbit of an individual particle is discussed briefly. *G. Randers.*

Chandrasekhar, S. The dynamics of stellar systems. IX–XIV. Astrophys. J. **92**, 441–642 (1940). [MF 3316]

Chandrasekhar continues a program started in an earlier publication [Astrophys. J. **90**, 1–154 (1939); these Rev. **1**, 60]. The problem is a purely mathematical one: to solve the "equation of continuity" for a stellar system. This equation, which must be satisfied by the distribution ψ of stars over space- and velocity-coordinates, is a homogenous, linear, partial differential equation in ψ. However, the gravitational potential is involved in the coefficients. An elimination of the potential (by Poisson's equation) would lead to a nonlinear, rather complicated integro-differential equation. As a first step toward the solution the author has therefore concentrated upon the linear equation, neglecting the Poisson equation altogether, leaving the gravitational potential as an unknown, but arbitrary function. Limiting himself to a special form of the distribution function (generalized Schwarzschild distribution) the author attacks the problem of determining corresponding forms of the potential and the distribution function, under further special assumptions about the one or the other. The present (second) part of this work is concerned with nonsteady states. The cases first considered are the equivalents of the steady states studied in the earlier publication, assuming circular, spherical and ellipsoidal equipotential surfaces. The calculations are extremely cumbersome, but lead to complete solutions of the numerous simultaneous equations involved.

The case of spherical symmetry of the distribution function in the residual velocities is studied in considerable detail. From the resulting general form of the spatial part of the distribution function the locus of points of constant density can be studied. The great variety of possible forms includes also different kinds of spiral structure.

Finally some remarks are made regarding the possibility of superposing two solutions ψ_1 and ψ_2 for the distribution function linearly. Considering the equation of continuity as linear in ψ, such superposition is possible. Since, however, the linearity is only apparent (ψ being involved in the coefficients through the potential), this procedure must be regarded with caution. *G. Randers* (Chicago, Ill.).

***Heuman, Carl. Ueber Bewegungsformen in globular clusters.** Astronomical Papers dedicated to Elis Strömgren, pp. 82–113. Einar Munksgaard, Copenhagen, 1940.

Elis Strömgren [Astr. Nachr. **203**, 17–24 (1916)] investigated the motion of a material particle in a globular cluster in which the density varies with the distance from the center r according to the law $\rho=\rho_0 a^5(a^2+r^2)^{-5/2}$, a being a constant. The central force is then

$$P=(4/3)\pi f m\rho_0 a^3 r(a^2+r^2)^{-3/2}.$$

The present author shows that this problem may be made to correspond to that of the spherical pendulum, and that the orbits of Strömgren's problem are obtained by central projection from the point of suspension of the pendulum on a horizontal plane.

Only orbits with $r_1\leqq r\leqq r_2$ are considered, r_1 and r_2 being finite; the limiting cases of rectilinear motion $r_1=0$ and of the pseudo-parabolic motion $r_2=\infty$ are included. The author, extending an earlier investigation [Bidrag till

teorien för sfäriska pendeln, Publ. Technolog. Inst. Stockholm 1918, 86 pp.], obtains the period from pericenter to pericenter, the motion of the apse and the arc described during this period in terms of elliptic integrals for the general case, and finds excellent agreement with Strömgren's results, secured by numerical integration.

D. Brouwer (New Haven, Conn.).

*Reiz, Anders. **On the fundamental equation of stellar statistics.** Astronomical Papers dedicated to Elis Strömgren, pp. 175–180. Einar Munksgaard, Copenhagen, 1940.

In stellar statistics one is generally concerned with the problem of deriving the frequency function of an "absolute" characteristic from the observed distribution of an "apparent" characteristic. The latter is obtained by effecting a suitable integration over the former taking into account the effect of distance, density variations and space absorption. [An example of such an integral relation is the one which expresses the true frequency of occurrence of stars of a given absolute magnitude in terms of the observed frequency function of apparent magnitudes.] In other words one deals with an integral equation of the form

$$\psi(z)=\int_{-\infty}^{+\infty}F(x)\phi(z-x)dx.$$

The author considers the case when $\phi(z-x)$ has the form

$$\phi(z-x)=\sum_{n=1}^{\infty}a_n\cos\left[2\pi p^{-1}n(z-x)\right]$$

and $F(x)$ can be expanded as a Charlier A-series

$$F(x)=\sum_{r=0}^{\infty}(-1)^r\frac{A_r}{r!}\frac{d^r}{dx^r}\left(\frac{e^{-x^2/2\sigma^2}}{\sigma(2\pi)^{\frac{1}{2}}}\right),$$

where the A_r's and σ are constants. Under these circumstances an explicit formula for $\psi(z)$ can be readily obtained and, as the author shows, we have

$$\psi(z)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi nz}{p}\right)e^{-(2\pi n\sigma/p)^2/2}\left[\sum_{s=0}^{\infty}\frac{A_{2s}}{(2s)!}\left(\frac{2\pi n}{p}\right)^{2s}(-1)^s\right]$$
$$+\sum_{n=1}^{\infty}a_n\sin\left(\frac{2\pi nz}{p}\right)e^{-(2\pi n\sigma/p)^2/2}$$
$$\times\left[\sum_{s=0}^{\infty}\frac{A_{2s+1}}{(2s+1)!}\left(\frac{2\pi n}{p}\right)^{2s+1}(-1)^{s+1}\right].$$

S. Chandrasekhar (Williams Bay, Wis.).

Hagihara, Yusuke. Electron velocity distribution in a planetary nebula. Jap. J. Astr. Geophysics **17**, 199–264 (1940). [MF 5821]

Hagihara, Yusuke. Electron velocity distribution in a planetary nebula. II. Jap. J. Astr. Geophysics **17**, 417–476 (1940). [MF 5822]

The mathematical interest of these papers is centered in the first one, of which the following is a review. This paper investigates the distribution of velocities in an assembly of free electrons under the conditions believed to exist in a planetary nebula. This problem differs from the standard problems of the kinetic theory of gases [see, for example, S. Chapman and T. G. Cowling, The Mathematical Theory of Non-uniform Gases, Cambridge University Press, Cambridge, England, 1939; these Rev. 1, 187] in that, in the case under consideration, the motions of the electron are influenced not only by its encounters with other electrons and ions but also by radiative processes involving captures and ionizations. These radiative processes are taken into account using the atomic probabilities for ionization and recombination according to the calculations of Gaunt [Philos. Trans. Roy. Soc. London. Ser. A. **229**, 163–204 (1930)]. Thus the discussion of the "encounter function" (which represents the net change in the number of electrons per unit time in any assigned velocity interval) has to be extended to include this new physical situation. The appropriate Boltzmann integro-differential equation is then written down. In solving this equation the author uses a method somewhat different from that customary in the kinetic theory of gases. Instead of expanding in terms of Sonine polynomials, in this paper the distribution function is expressed as a convergent series of Hermite polynomials. Similarly, the encounter function is also expanded in terms of these polynomials. These series are then substituted in the Boltzmann equation and the coefficients of the different orders of the Hermite polynomials are equated. This is the manner in which the author seeks the solution to the problem. Several special cases are considered and applications to the problem of the time of relaxation are also sketched.

S. Chandrasekhar (Williams Bay, Wis.).

Randers, Gunnar. The equilibrium and stability of ring-shaped "barred spirals." Astrophys. J. **95**, 88–111 (1942). [MF 6282]

This paper studies the equilibrium and stability of an incompressible liquid ring (or torroid) rotating with a uniform angular velocity ω about a central body of mass M_c. The discussion is carried out for the case where the diameter of the torroid (D) is large compared to the diameter of its cross section.

I. It is a result of Laplace that in equilibrium the ring has an elliptic cross section with axes a and b such that

$$(1)\qquad \Omega=\frac{\omega^2}{\pi G\rho}=\frac{M_c}{\pi\rho D^3}=\frac{4ab(a-b)}{(3a^2+b^2)(a+b)},$$

where ρ is the density of the ring and G the constant of gravitation. A discussion of the foregoing equation shows that

$$(2)\qquad \Omega\leqq 0.2172.$$

The interpretation of this result is that for each distance from the center (greater than a certain critical distance) there are two configurations possible, one of which is more elliptical than the other. At the critical distance the two sequences coalesce to give a unique configuration with a ratio of axes $b/a=0.3855$. For distances less than the critical distance no equilibrium configurations are possible.

II. The stability of the ring is next considered for four different types of perturbations: (i) A uniform expansion and contraction (that is, changes in D) with homologous changes in a and b (that is, a/b constant). Such perturbations are shown to lead to stable oscillations. (ii) Perturbations causing the area of the elliptic cross section to vary along the circular axis of the torroid in such a way that D and the ratio a/b remain constant. It is shown that instability sets in first, for variations of a and b according to

$$(3)\qquad a=a_0(1+A\cos\phi),\quad b=b_0(1+A\cos\phi),\qquad A\ll 1,$$

where ϕ is the azimuthal angle measured along the circular

axis. The corresponding condition for stability is

$$\frac{M_c}{M_R} > \frac{1}{2\pi}\left(\log\frac{16D}{a_m+b_m}-\frac{15}{4}\right), \tag{4}$$

where

$$a_m^2=a_0^2(1+\tfrac{1}{2}A^2),\quad b_m^2=b_0^2(1+\tfrac{1}{2}A^2), \tag{5}$$

and M_c and M_R denote the mass of the central body and the ring, respectively. This result (4) is a generalization of a result due to Dyson [Philos. Trans. Roy. Soc. London. Ser. A. 184, 1041 (1893)], who considered the stability of a torroid without a central body. According to (4), for $M_c=M_R$ instability requires $D\sim 10^4 a_m$; in other words, for all practical purposes the ring may be considered as stable for such perturbations. (iii) Perturbations causing a variation of D with the azimuth ϕ, while other things remain constant. For such perturbations, the less elliptical of the two possible equilibrium configurations [see I above] is stable, while the more elliptical is unstable; these latter configurations therefore go over into the former. (iv) Finally the author considers perturbations similar to case (ii) above with the difference, however, that the variations in a and b are now assumed not to be fixed in space, but to follow the rotation of the ring. For such perturbations the ring is shown to be unstable. *S. Chandrasekhar.*

Cowling, T. G. The non-radial oscillations of polytropic stars. Monthly Not. Roy. Astr. Soc. **101**, 367–375 (1942). [MF 6290]

The gravitational interaction between two stars in a binary give rise to periodic perturbations on each of the stars. To investigate whether there is any possibility of resonance between such tidal disturbances and free oscillations of stars, the author studies non-radial oscillations of a polytropic star. Assuming first the gravitational potential to remain unaffected, he obtains two first order equations for the material displacement and the pressure variation (both assumed to be products of surface harmonics and functions of radius). An infinite number of extremely long and extremely short periods are found to exist. The effect on the periods of leaving out the variation in gravitational potential is found to be small for a polytrope of index 3.

Because of the existence of very long periods of the free oscillations, resonance with the tidal disturbance is possible. Because of the nature of the free oscillations in question, however, it is concluded that they will probably not affect the tidal distortion to any great extent. The problem of the origin of the moon by resonance fission of the earth is discussed in this connection. *G. Randers.*

Bhatnagar, P. L. Polytropic gas models with variable angular velocity. Bull. Calcutta Math. Soc. **32**, 21–41 (1940). [MF 6154]

This paper deals with the problem of the slow rotation of polytropic gas spheres in which the angular velocity ω depends on the distance from the center (r) and the cosine of the colatitude (μ) according to the equation

$$\omega^2=\omega_0^2\{1-\alpha r^2 R_s^{-2}(1-\mu^2)\}, \tag{1}$$

where ω_0, $\alpha(<1)$ and R_s ($<R_1$, the radius of the gas sphere) are all constants. The problem is solved by standard perturbation methods: For no rotation the distribution of the density (ρ) in polytropic gas spheres is governed by the Lane-Emden differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n, \tag{2}$$

where ξ is radius vector measured in appropriate units and $\rho=\rho_c\theta^n$, ρ_c being the central density. For a rotating star the corresponding differential equation to be solved is

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-\Theta^n+\frac{\omega^2}{2\pi G\rho_c}. \tag{3}$$

For slow rotation, we can write

$$\Theta=\theta+\frac{\omega^2}{2\pi G\rho_c}\Phi+O(\omega^4), \tag{4}$$

and further expand Φ in a series involving purely radial functions and the Legendre polynomials $P_j(\mu)$. For the case when ω is variable but depends on r and μ according to (1), the author shows that the appropriate solution satisfying the necessary boundary conditions is

$$\begin{aligned}\Phi=\phi_0(\xi)&-\frac{5}{6}\frac{\xi_1^2}{3\phi_2(\xi_1)+\xi_1\phi_2'(\xi)}\phi_2(\xi)P_2(\mu)\\ &+\frac{4}{3}\frac{\alpha}{\xi_s^2}\Bigg[f_0(\xi)+\left\{f_2(\xi)-\frac{3f_2(\xi_1)+\xi_1 f_2'(\xi_1)-\frac{1}{2}\xi_1^4}{3\phi_2(\xi_1)+\xi_1\phi_2'(\xi_1)}\phi_2(\xi)\right\}P_2(\mu)\\ &-\frac{27}{140}\frac{\xi_1^4}{5\phi_4(\xi_1)+\xi_1\phi_4'(\xi_1)}\phi_4(\xi)P_4(\mu)\Bigg],\end{aligned} \tag{5}$$

where ϕ_0, ϕ_2, ϕ_4, f_0 and f_2 are solutions of certain second order differential equations which vanish at $\xi=0$ and have further vanishing first derivatives at this point. In equation (5), ξ_1 denotes the first zero of $\theta(\xi)$, and a prime indicates differentiation with respect to ξ.

The functions ϕ_0, ϕ_2 and ϕ_4 have already been integrated by Chandrasekhar in his studies on distorted polytropes [Monthly Not. Roy. Astr. Soc. **93**, 390–405, 449–471, 539–574 (1933)] and the author now provides corresponding tables of solutions for the functions f_0 and f_2 for the values of $n=1$, 1.5, 2, 3 and 4. *S. Chandrasekhar.*

Hydrodynamics, Aerodynamics, Acoustics

Dolidze, D. Boundary problems of the linear hydrodynamical equations in the plane and the symmetrical space. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **7**, 65–103 (1940). (Russian. English summary) [MF 5304]

The author considers the first boundary problem for the two-dimensional unsteady motion of a viscous incompressible fluid, that is, the problem of finding in a domain D the solutions $v_x(x, y, t)$, $v_y(x, y, t)$ of the Navier-Stokes equations for slow motion:

$$\mu\Delta v_x-\frac{\partial v_x}{\partial t}=\frac{\partial q}{\partial x},\quad \mu\Delta v_y-\frac{\partial v_y}{\partial t}=\frac{\partial q}{\partial y},\quad \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,$$

where q denotes the unknown pressure p (or $p+v^2/2$). The initial and boundary conditions are: (1) $v_x(x, y, 0)=v_y(x, y, 0)=0$ throughout D and (2) v_x, v_y are given functions on the boundary of D for all values of the time t. The author has obviously overlooked the fundamental

papers of J. Leray [J. Math. Pures Appl. (9) **13**, 331–418 (1934), especially pp. 353–363; Enseignement Math. **35**, 139–151 (1936) (Conférences Internationales des Sciences Mathématiques)] in which the same problems have been treated and solved under appropriate restrictions. Though the author's method is from the formal viewpoint not essentially different from Leray's, the existence and unicity theorems are seemingly obtained without being hindered by any of the difficulties disclosed by Leray's analysis. The author considers also the axial symmetrical flow.

A. Weinstein (Toronto, Ont.).

Dolidze, D. Das allgemeine lineare Randwertproblem der Hydrodynamik. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 43–49 (1941). (Russian. German summary) [MF 5297]

The author considers the non-stationary motion of an incompressible viscous fluid in an interior or exterior region, which is bounded by a regular closed curve. In the case of a fixed region, the linearized equations of hydrodynamics are integrated for preassigned initial and boundary values of the velocity. Next, the so-called fundamental tensor is constructed, and by means of this the problem is reduced to the solution of a system of three quasi-regular integral equations of the Volterra type. By applying the usual theory to this system, one can then construct the unique set of solutions.

A. Gelbart (Raleigh, N. C.).

Dolidze, D. Über ein nichtlineares Problem der Hydrodynamik. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 659–664 (1940). (Russian. German summary) [MF 5287]

The plane boundary value problem of the non-stationary motion of an incompressible viscous liquid for a fixed region, bounded by a regular closed curve, is considered. A method is given to find the stream function Ψ when the initial and boundary values of the velocity are known. The solution of this leads to a system of nonlinear integral equations, which, in the case of an interior region, can be solved by the method of successive approximation. For this process a convergence criterion is given which shows the dependence of the convergence on the shape and dimensions of the region and on the velocity and the coefficient of viscosity.

A. Gelbart (Raleigh, N. C.).

Kravtchenko, Julien. Sur le problème de représentation conforme de Helmholtz; théorie des sillages et des proues. J. Math. Pures Appl. (9) **20**, 35–234 (1941). (To be continued) [MF 4639]

Consider the steady two dimensional irrotational flow of a non-viscous incompressible fluid in the z-plane between two infinitely long straight parallel walls μ_1, μ_2 containing a given body specified by the curve BOC; B and C are the terminal points and O is the point of bifurcation of the streamline. At two points on BOC the flow separates from the body and forms two free streamlines where the velocity is constant. If the points of separation are given, the problem of determining the flow is called the wake problem. If the points of separation are unknown except that they must lie on BOC, but the flow is subject to two auxiliary conditions due to Brillouin [Ann. Physique Chemie (8) **23**, 145–230 (1911), in particular, p. 154], the problem is called the bow problem. In the first chapter, the previous work of T. Levi-Civita and H. Villat is reviewed in detail. The chief concern here is to represent conformally the flow in the z-plane within an annular region in the Z-plane between $Z=\epsilon^{is}$ and $Z=qe^{is}$, $0\leqq s\leqq\pi$. It is difficult, however, to obtain the conformal transformation explicitly in terms of the given boundary in the z-plane. Thus a function $\psi(s)$ is specified which gives the angle formed by the walls μ_1, μ_2 and BOC in the z-plane at the corresponding point of $Z=\epsilon^{is}$ in the Z-plane. The solution in the Z-plane can then be written explicitly in terms of $\psi(s)$ which is subject to certain conditions.

In the second chapter, the solution so specified is examined from the standpoint of Brillouin's auxiliary conditions. For the case of infinitely large distance between μ_1, μ_2, J. Leray has shown that, with certain restrictions on $\psi(s)$, this solution also satisfies the auxiliary conditions. These restrictions on $\psi(s)$ require, among other things, that lines parallel to μ_1, μ_2 only intersect BOC once and that the only discontinuity of the tangent of BOC occurs at O. The author proved this theorem for flow bounded by the walls μ_1, μ_2. In the third chapter, the existence of at least one solution for both the wake problem and the bow problem is proved under the assumption that $0\leqq\psi(l)\leqq\pi$, $d\psi(l)/dl$ exists and is continuous for $\alpha\leqq l\leqq\beta$, where l is the length along the curve BOC. For this purpose, the author made use of a certain theorem about functional equations obtained by Leray and Schauder [Ann. École Norm. (3) **51**, 45–78 (1934)]. If one of the walls approaches the obstacle and finally touches C, the point of bifurcation O and C approach each other and finally O coincides with C. Furthermore, the velocity at every point in the field is shown to converge uniformly to the final value, except at the point C.

H. S. Tsien (Pasadena, Calif.).

Munk, Max M. On some vortex theorems of hydrodynamics. J. Aeronaut. Sci. **9**, 90–96 (1942). [MF 6097]

The vortex theorems and the equations for the kinetic energy, the moment and the moment of momentum of an incompressible fluid are derived. The action of external forces on the motion, the physical meaning of the different theorems and the relations between the differential and integral form of the equations are thoroughly discussed.

H. W. Liepmann (Pasadena, Calif.).

Van Mieghem, Jacques. Contribution à l'étude du mouvement de l'air dans les perturbations d'altitude associées aux ondes du front polaire. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **19**, fasc. 3, 65 pp. (1941). [MF 5834]

The author considers the approximate equations of motion of the atmosphere

$$\begin{aligned}
&\frac{dv_x}{dt}-\lambda v_y=-s\frac{\partial p}{\partial x}+s\eta\frac{\partial^2 v_x}{\partial z^2},\\
&\frac{dv_y}{dt}+\lambda v_x=-s\frac{\partial p}{\partial y}+s\eta\frac{\partial^2 v_y}{\partial z^2}, \qquad \lambda\equiv 2\omega\sin\varphi, \qquad (1)\\
&g=-s\frac{\partial p}{\partial z},\\
&\frac{\partial}{\partial x}\left(\frac{v_x}{s}\right)+\frac{\partial}{\partial y}\left(\frac{v_y}{s}\right)+\frac{\partial}{\partial z}\left(\frac{v_z}{s}\right)=0.
\end{aligned}$$

v_x, v_y are the horizontal, v_z the vertical component of the wind velocity; s and p are the specific volume and the pressure; η denotes a friction coefficient; ω and φ denote the angular velocity of the earth and the latitude. Starting from

the expression for the geostrophic wind

$$(2)\qquad v_x^{(0)}=-\frac{s}{\lambda}\frac{\partial p}{\partial y},\quad v_y^{(0)}=+\frac{s}{\lambda}\frac{\partial p}{\partial x},\quad v_z^{(0)}=0,$$

an iteration method, that is,

$$(3)\qquad v_x^{(n+1)}=v_x^{(0)}-\frac{1}{\lambda}\frac{d^{(n)}v_y^{(n)}}{dt}+\frac{s}{\lambda}\eta\frac{\partial^2 v_y^{(n)}}{\partial z^2}$$

and the corresponding equations for the v_y, is applied to (1); here

$$\frac{d^{(n)}}{dt}=\frac{\partial}{\partial t}+v_x^{(n)}\frac{\partial}{\partial x}+v_y^{(n)}\frac{\partial}{\partial y}.$$

If only the horizontal components of the wind are considered, (2) and (3) furnish in first approximation

$$(4)\qquad \bar{v}=\bar{v}^{(0)}-\frac{1}{\lambda}\frac{d^{(0)}}{dt}\left(\frac{s}{\lambda}\nabla p\right)+\frac{s}{\lambda^2}\eta\frac{\partial^2}{\partial z^2}(s\nabla p).$$

Several special cases of (4) leading to known approximate expressions for the wind are discussed. The author then applies (4) to the motion in the depression connected with a cyclone. Putting $p=P+p^*$, etc., where P corresponds to an equilibrium state and p^* denotes the pressure due to a small perturbation, (4) is solved neglecting friction. The results are shown to be in agreement with meteorological observations and especially the "convergence-divergence effect" of J. Bjerknes follows quantitatively from this analysis. The degree of approximation reached with (4) is given numerically. By comparing the results of the perturbation method starting from (4) and from the well-known expression for the wind of Brunt and Douglas the improvement due to the approximation method (2) is pointed out.

H. W. Liepmann (Pasadena, Calif.).

Jacob, Caius. Sur l'écoulement lent d'un fluide parfait, compressible, autour d'un cylindre circulaire. Mathematica, Cluj **17**, 1–18 (1941). [MF 5550]

Chaplygin's approximate method [Ann. Sci. Univ. Moscow **1904**, 1–121] has been devised for the jet theory. Slioskin extended this method to the case of continuous motion in two dimensions [C. R. (Doklady) Acad. Sci. URSS (N.S.) **3**, 419–421 (1936)]. The integro-differential equation to which this method leads in the case of the flow around a circular obstacle is studied with special reference to the questions of existence and uniqueness of solution.

W. Prager (Providence, R. I.).

von Kármán, Th. Compressibility effects in aerodynamics. J. Aeronaut. Sci. **8**, 337–356 (1941). [MF 5330]

The aim of the paper is to "review the present status of the theory of compressible fluids from the practical standpoint of its usefulness for interpretation of experimental research and guidance in design." First it is explained that irrotational flow of compressible fluids is not a matter of pure geometry as it is for compressible fluids. The thermodynamic relation between density and pressure must be taken into account; it yields, in particular, the "velocity of sound." To illustrate the basic relations analogies are considered: the flow in inclined channels, the current in an electrolytic bath. The one-dimensional or "hydraulic" approximate treatment of compressible fluid flow in a converging-diverging nozzle reveals a most striking feature: if the velocity of sound is exceeded, there must be a sudden transition from supersonic to subsonic velocity, a "shock," somewhere in the diverging part of the nozzle.

"The main problem of primary interest for aeronautical application is that of the stationary flow around a body." It is a boundary value problem: to determine a velocity potential $\varphi(x_1, x_2, x_3)$ such that

$$\left[a_0^2-c\sum_{k=1}^{3}\left(\frac{\partial\varphi}{\partial x_k}\right)^2\right]\sum_{i=1}^{3}\frac{\partial^2\varphi}{\partial x_i^2}=\sum_{i,k=1}^{3}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_k}\frac{\partial^2\varphi}{\partial x_i\partial x_k},\quad a_0, c \text{ constants},$$

outside the body while on its boundary the normal derivative vanishes, and $\partial\varphi/\partial x_1=U$, $\partial\varphi/\partial x_2=\partial\varphi/\partial x_3=0$ at infinity. Various approximation methods are explained: expansion with respect to Mach's number $M=U/a_0$; a similar perturbation method applicable to thin airfoils. Chaplygin's hodograph method consists in considering the velocity components $\partial\varphi/\partial x_i$ as independent variables. The differential equation then becomes linear with non-constant coefficients. These coefficients can be so approximated by constants in several ways that the equation reduces to Laplace's equation. Exact solutions can also be obtained in special cases. If Mach's number exceeds a "critical" value it happens that re-introduction of the variables x_i is impossible due to vanishing Jacobian. Then a shock must appear. Such shocks and resultant "compressibility bubbles" are observed experimentally; they are held responsible for a sudden drop in lift. Methods for determining Mach's number are discussed.

Completely supersonic flow is not considered in the paper. A great many results derived from the approximation methods explained are confronted with experimental results in detail. The report closes with an extensive bibliography.

K. Friedrichs (New York, N. Y.).

Weyl, Hermann. Concerning the differential equations of some boundary layer problems. Proc. Nat. Acad. Sci. U. S. A. **27**, 578–583 (1941). [MF 5908]

The author considers first the differential equation for the boundary layer in the flow of a viscous incompressible fluid with constant velocity past the half-plane $y=0$, $x\geqq 0$, which can be reduced to (1) $w'''+2ww''=0$ $(0\leqq z<\infty)$. He shows that the commonly-used solution of H. Blasius [Z. Math. Phys. **56**, 1 (1908)] becomes divergent somewhere between $z=\sqrt[3]{9}$ and $\sqrt[3]{30}$, and proposes a new solution based on the fact that (1) with $w=w'=0$, $w''=1$ for $z=0$ is equivalent to the integral equation

$$g(z)=\exp\left(-\int_0^z(z-\zeta)^2 g(\zeta)\,d\zeta\right)\quad\text{or}\quad g=\Phi\{g\},$$

where $g(z)=f''(z)$. The properties of Φ lead to a solution by successive approximations: $g_{n+1}=\Phi\{g_n\}$, $n=0, 1, \ldots$, starting with $g_0(z)=0$. The process is shown to converge rapidly, and there is evidence that $g_3(z)$ is a pretty good approximation throughout the entire interval. With slight alterations this process is also applied to the equation for the wake behind the flat plate

$$w'''+2ww''-w'^2=0$$

and to the equation for the three-dimensional case of flow against a perpendicular wall

$$w'''+2ww''-(V^2-w'^2)=0.$$

W. R. Sears (Inglewood, Calif.).

Weyl, Hermann. Concerning the differential equations of some boundary layer problems. II. Proc. Nat. Acad. Sci. U. S. A. **28**, 100–102 (1942). [MF 6331]

In an earlier note [see the preceding review] the author solved by successive approximations the boundary value problem $w'''+2ww''+2\lambda(k^2-w'^2)=0$ for $z\geqq 0$, $w=w'=0$ for $z=0$, $w'(\infty)=k>0$ for the special values $\lambda=0$ and $\frac{1}{2}$. He now extends the work for arbitrary $\lambda\geqq 0$, using the method of fixed points of functional transformations inaugurated by G. D. Birkhoff and Kellogg [Trans. Amer. Math. Soc. **23**, 96–115 (1922)]. A functional operation $\varphi=\Phi_\lambda\{g\}$ is set up; the nature of this operation is such that the boundary value problem is solved if a function g can be found that is equal to its image φ. The existence of such a "fixed point" is established by use of a theorem of G. D. Birkhoff and Kellogg. *W. R. Sears.*

Kolmogoroff, A. N. Dissipation of energy in the locally isotropic turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 16–18 (1941). [MF 5847]

For locally isotropic turbulence as defined in an earlier paper [same C. R. **30**, 301–305 (1941); these Rev. **2**, 327], the author states without deduction a differential equation relating the energy dissipation with the double and triple correlations of velocities. Integration of this equation leads to conclusions regarding the behavior of the triple correlation for large distances. Asymptotic formulas are also deduced for the double correlations for distances small compared to L, the integral scale of turbulence [e.g., Kolmogoroff, same C. R. **31**, 538–540 (1941); these Rev. **3**, 26], but large compared to λ, the local scale. These formulas contain one constant C, which is dependent only on the final value of a ratio involving double and triple correlations. The formulas are applied to experimental results of Dryden, et al [Nat. Adv. Comm. Aeronaut., Rep. No. **581** (1937)], and good agreement is found for three different mesh sizes for a single value of C. *W. R. Sears.*

Obukhoff, A. On the energy distribution in the spectrum of a turbulent flow. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 19–21 (1941). [MF 5848]

The spectral decomposition of a random function of one variable has been pointed out by Kolmogoroff [C. R. Acad. Sci. URSS (N.S.) **26**, 6–9, 115–118 (1940); cf. these Rev. **2**, 220]. The generalization to the three-dimensional case is given by the author in the form of a multiple Fourier-Stieltjes integral taken over the three-dimensional space of "frequencies." Here the random function is the field of velocities of a turbulent flow. The distribution function of energy $\varphi(p)$ is calculated in terms of this formulation. For any given "observation scale" l and "observation frequency" $p=1/l$, the field of velocities is decomposed into two orthogonal components, the "macrocomponent" and the "microcomponent," by carrying out the integration inside and outside a sphere of radius p in the space of frequencies. The energy balance of the microfield involves dissipation and the transfer of energy from the macrofield; these quantities are evaluated in terms of $\varphi(p)$. An expression for the energy distribution $E(p)$ in the low frequencies is obtained. An extension is made to the case of flow bounded by a scale L. The author states that a more detailed exposition will be published in the Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. *W. R. Sears* (Inglewood, Calif.).

Schumann, T. E. W. An investigation concerning G. I. Taylor's correlation coefficient of turbulence. Philos. Mag. (7) **32**, 471–482 (1941). [MF 6046]

The correlation coefficient R_ξ between turbulent velocities at times t and $t+\xi$ at a fixed point in a fluid was introduced by Taylor [Proc. London Math. Soc. (2) **20**, 196 (1921)]. The present author objects to Taylor's proof that the expansion for R_ξ contains only even powers of ξ, on the basis that the distribution function extends to infinity in both positive and negative directions and therefore the derivatives with respect to time involved in Taylor's proof may also assume infinite values. He deduces an expression for R_ξ in terms of the distribution function of velocities $F(p)$ and the function $f(p, \xi, u)$ which is the probability that after a time ξ the velocity of a particle whose original velocity lay between p and $p+dp$ lies between u and $u+du$. If $F(p)$ is known, the calculation of the correlation coefficient depends on one's ability to determine $f(p, \xi, u)$.

The author considers, in particular, cases in which

$$F(p)=\alpha^{\frac{1}{2}}\pi^{-\frac{1}{2}}e^{-p^2/\alpha^2},$$

where $\alpha=$constant. This is analogous to the final distribution of temperature in a certain heat-transfer problem, and on this basis the author adopts as the differential equation for $F(p)$ the equation of this problem,

$$\frac{\alpha^2}{k}\frac{\partial F}{\partial t}=\alpha^2\frac{\partial^2 F}{\partial p^2}+2p\frac{\partial F}{\partial p}+2F,$$

which is solved in terms of exponential and Hermite functions. In the special case of a unit source at p at $t=0$, the function $f(p, \xi, u)$ is evaluated and its numerical values are plotted for three values of $\xi k/\alpha^2$. In this case R_ξ can be calculated exactly from the expression previously deduced; it is found to be $R_\xi=e^{-2k\xi/\alpha^2}$, which does not conform to Taylor's result. Finally, experimental results for R_ξ obtained from daily minimum temperature readings are compared with the corresponding values of the theoretical curve which best fits the data. *W. R. Sears* (Inglewood, Calif.).

Wieghardt, K. Correlation of data on the statistical theory of turbulence. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1008**, 20 pp. (1942). (3 plates) [MF 6291]

Translation of a paper in Luftfahrtforschung **18**, 1–7 (1941); cf. these Rev. **2**, 327.

Southwell, Richard V. New pathways in aeronautical theory. J. Aeronaut. Sci. **9**, 77–87; discussion, 87–89 (1942). [MF 6096]

The philosophical background of the author's "relaxation" method of successive approximations [Relaxation Methods in Engineering Science, Oxford University Press, New York, 1940; these Rev. **3**, 152] is discussed. The scope of the method is indicated by illustrations of the types of problems handled by it. These include vibration problems, plane-potential problems involving arbitrary boundaries and several problems considered practically insolvable by more common methods: torsion of a bar partially into the plastic regime, determination of magnetic lines in fields containing iron and percolation through a dam with flow down the rear face. The mathematical details of the technique are not given in this paper. *W. R. Sears.*

Biot, M. A. Some simplified methods in airfoil theory. J. Aeronaut. Sci. **9**, 185–190 (1942). [MF 6335]

It is shown how, by the combined use of conformal mapping and L. Prandtl's acceleration potential, the solution

of certain problems in two-dimensional airfoil theory can be greatly simplified. The boundary conditions for the acceleration potential are satisfied, not along the actual contour of the thin airfoil, but along both sides of its projection onto the real axis which has the direction of the velocity of the undisturbed flow. Mapping the plane which is slit along this portion of the real axis onto the exterior of a circle, the author obtains a boundary value problem whose solution, in the examples given in the paper, can be built up from simple singularities on or in the interior of this circle. The following examples are discussed: the symmetric airfoil with flap, the airfoil giving a uniform lift-distribution, the oscillating airfoil. *W. Prager.*

Burington, Richard S. and Dobbie, James M. A new family of wing profiles. J. Math. Phys. Mass. Inst. Tech. **20**, 388–401 (1941). [MF 6108]

The region to the left of the right hand branch of the hyperbola $x^2/a^2-y^2/b^2=1$ is mapped into the region exterior to a closed curve resembling an airfoil section by the transformation $\zeta=e^{i\beta}/(z-z_0)$, where $z_0=x_0+iy_0$ is a suitably chosen point to the left of the considered branch of the hyperbola and $\tan\beta=-y_0/(a-x_0)$. A two-parametric family of airfoil sections is thus obtained and the transformation mapping the region exterior to any one of these curves into the region outside the unit circle is given. *W. Prager.*

Loring, S. J. General approach to the flutter problem. SAE J. **49**, 345–355; discussion, 355–356 (1941). [MF 5040]

For the purpose of this paper flutter is defined as a self-excited oscillation arising out of the simultaneous action of elastic, inertia and aerodynamic forces upon a system of masses. The developments presented are based on the equations of motion in the form of Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial V}{\partial q_i}=Q_i, \qquad i=1, 2, \cdots, r,$$

where T is the kinetic energy of the system, V the potential energy and the Q_i the generalized external (aerodynamic) forces. The author expresses the displacements of the system in the form

$$(u_1, u_2, u_3)=\sum_{i=1}^{r} q_i(t)(\phi_{1i}(x_1, x_2, x_3), \phi_{2i}(x_1, x_2, x_3), \phi_{3i}(x_1, x_2, x_3)),$$

the q_i being generalized displacements and the ϕ_i generalized coordinates. The important step consists in representing the actual motion closely by means of a very few suitably chosen functions ϕ_i. Then the kinetic energy and the potential energy are calculated as quadratic forms in the $\dot{q}_i$ and q_i, respectively, and these are introduced into the equations of motion together with expressions for the airforces of the form

$$Q_i=\sum_{j=1}^{r}(A_{ij}q_j+B_{ij}\dot{q}_j+C_{ij}\ddot{q}_j),$$

the coefficients A, B, C depending on airspeed U and on the geometry of the airfoil in a manner to be determined by aerodynamics. There result then r simultaneous linear differential equations with constant coefficients for the q_i. Flutter occurs if these equations have oscillatory solutions of the form $q_i=(D_i'+iD_i'')e^{i\omega t}$, which when introduced into the equations of motion lead to simultaneous, homogeneous linear equations for the complex coefficients $(D_i'+iD_i'')$. Vanishing of the complex determinant of this system leads to two simultaneous real equations for the flutter frequency ω_f and the flutter speed U_f.

As an example the author sketches the treatment of the problem of a wing plus aileron, the wing subject to bending and torsion, the aileron subject to bending and a deflection about its hinge line, the airforces being those obtained when the known expressions of the two-dimensional theory [Theodorsen, Küssner] are taken as section forces for the wing of finite span. *E. Reissner* (Cambridge, Mass.).

Lewis, T. Some properties of the hodograph and their application in the construction of the hodograph in the problem of external ballistics. Philos. Mag. (7) **32**, 427–435 (1941). [MF 6044]

For the case in which the variation of resistance with altitude is neglected the author establishes the following three properties of the hodograph associated with the trajectory of a projectile: (1) Hodographs are orthogonal trajectories of a certain one-parameter family of curves. These curves are closed curves and the hodographs converge to the limiting point of these curves. (2) The tangents to all hodographs at points corresponding to a prescribed velocity v pass through a fixed point provided the ballistic coefficient is constant. (3) The area swept out by the radius vector from the origin to the hodograph is proportional to the horizontal distance travelled by the projectile. These properties furnish a relatively simple graphical method for constructing the hodographs for any law of resistance $f(v)$ given by tables, provided the ballistic coefficient is constant. From the hodograph the trajectory may be obtained by numerical or graphical integrations. For the case in which the variation in air density is not ignored the problem can still be handled graphically with about the same labor, but of course the hodograph cannot in this case be constructed independently of the trajectory. The two must be drawn simultaneously step by step. *W. E. Milne.*

McShane, E. J. Computation of flat trajectories with high angles of departure. Amer. Math. Monthly **48**, 617–623 (1941). [MF 5896]

The method presented is similar in spirit to that of Siacci. It employs a "pseudo-velocity" w which is proportional to the projection of the velocity v upon a line with direction about halfway between the initial and final directions of v. This pseudo-velocity w is expressible linearly in terms of x' and y'. By quadratures and successive approximations, one obtains expressions for the coordinates, velocity components and time of flight. The variation of air density with altitude is not ignored. *W. E. Milne* (Corvallis, Ore.).

d'Adhémar, Robert. Le mouvement gyroscopique des projectiles. L'axe dynamique d'équilibre de M. Ernest Esclangon. C. R. Acad. Sci. Paris **211**, 584–585 (1940). [MF 5365]

Fock, V. A. A theoretical investigation of the acoustical conductivity of a circular aperture in a wall put across a tube. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 875–878 (1941). [MF 5517]

Consider an infinite circular tube of radius a with a transverse thin wall pierced by a circular aperture of radius α with center on the tube's axis. The velocity potential is $\phi=v+bz$, where b is constant and v is an everywhere finite odd harmonic function. The boundary conditions are (1) $\partial v/\partial r=0$ for $r=a$, $v=0$ for the aperture and $\partial\phi/\partial z=0$ for

the wall. The parameter of key interest is $v_0=v(\infty)$. The solution for v may be written

$$(2) \qquad v=v_0+\sum_1^\infty v_s \exp -\lambda_s z/a J_0(\lambda_s r/a),$$

where $J_1(\lambda_s)=0$. In principle, v_s and v_0 may be determined by substituting (2) in (1) and solving the infinite system of linear equations so obtained. However, the author develops a "practical" solution method by representing the association of (1) and (2) by an equivalent linear integral equation involving v_0. The integral equation then yields a solution for v_0 as a power series in α/a. *D. G. Bourgin.*

Landahl, H. D. A kinetic theory of diffusion forces in metabolizing systems. Bull. Math. Biophys. **4**, 15–26 (1942). [MF 6289]

Estimates of diffusion "drag" forces are required in the consideration of cellular reproduction and of cellular deformations in general. Rashevsky [Mathematical Biophysics, University of Chicago Press, Chicago, Ill., 1938] and Young [Growth **2**, 165–180 (1938)] concluded that these forces acting on spherical particles are proportional to the cube of the radius (that is, to the volume). The force of the diffusing solvent (water) was neglected. The author employs the theory of non-uniform gases to obtain an expression involving also the second power of the radius, according to which, under certain circumstances, the forces due to the solvent molecules may dominate. This is for small spheres. For larger spheres a hydrodynamical model gives the force proportional to the first power of the radius and again a possible dominance by the solvent. *A. S. Householder.*

Theory of Elasticity

Tonolo, A. Alcune analogie fra la geometria delle varietà riemanniane a tre dimensioni e la meccanica dei mezzi continui. Boll. Un. Mat. Ital. (2) **3**, 353–359 (1941). [MF 5700]

The analogies between Riemannian geometry in three dimensions and the statics of a continuous medium rest on the facts that (a) the Ricci tensor and the stress tensor are of the same type, and (b) the Bianchi identities satisfied by the Ricci tensor agree formally with the conditions of equilibrium satisfied by the stress tensor in the absence of body forces. The following is one of the results of the paper: If a continuous medium (not under the influence of body forces) is distributed in a Riemannian space in such a way that the covariant components of the stress tensor are equal at each point to those of the Ricci tensor, then the medium is in equilibrium under the action of its internal forces alone. *J. L. Synge* (Toronto, Ont.).

Locatelli, Piero. I più generali riflessi della congruenza delle deformazioni sugli sforzi elastici. Pont. Acad. Sci. Acta **5**, 73–83 (1941). [MF 5888]

The author investigates the general conditions which have to be satisfied by a doubly symmetrical tensor if it can be considered as stress tensor of an elastic continuum. One of the main results is that, if the continuum is two-dimensional, isotropic and Euclidean, no such restricting condition exists; in other words, any doubly symmetrical tensor can be considered as stress tensor of the two-dimensional continuum provided that not only the field-force-components but also the two elastic moduli are chosen as suitable functions of the two coordinates. *P. Nemenyi.*

Friedrichs, K. O. and Stoker, J. J. The non-linear boundary value problem of the buckled plate. Amer. J. Math. **63**, 839–888 (1941). [MF 5629]

The buckling of a plate for loads beyond the lowest critical load is described by a non-linear boundary value problem formulated by von Kármán. The authors make a significant contribution to this problem from the viewpoints of both pure and applied mathematics by a detailed and rigorous treatment of the case of a circular plate. Assuming that the deflection w and the Airy stress function φ depend on r alone ($0\leqq r\leqq 1$), the authors introduce the new functions p and q, proportional to $r^{-1}\varphi_r$ and $-r^{-1}w_r$, respectively, and write von Kármán's equations as follows: $Gp=\frac{1}{2}q^2$, $Gq=-pq$, where $G=r^{-3}d(r^3d/dr)/dr$, the boundary conditions being $p_r=q_r=0$ for $r=0$; $p=\lambda^2$, $q_r+(1+\nu)q=0$ for $r=1$. For a given Poisson's ratio ν, the problem depends only on one parameter, the given boundary thrust, represented by λ^2. In part I (viewpoint of applied mathematics) the numerical solution is worked out for the entire range $0\leqq\lambda^2\leqq\infty$ by three methods, each suitable for a particular range: perturbation method, power series method, asymptotic solution ($\lambda^2=\infty$). The asymptotic solution is necessary in order to clarify the striking occurrence of a boundary layer effect; while with increasing λ^2 all quantities tend to become constant in the interior of the plate, they change rapidly in a narrow strip at the boundary. This edge effect is treated by a stretching of the scale in such a manner that the edge strip does not shrink to zero with increasing λ^2.

In part II (viewpoint of pure mathematics) the assumptions made in part I are justified by an investigation of the existence and uniqueness of the solutions of the problem and their continuous dependence on λ^2. The boundary problem is found to be closely related to the following minimum problem: $qDq-\lambda^2qHq+p_rKp_r=\min.$, where $qDq-(1+\nu)q^2(1)$, qHq and p_rKp_r denote, respectively, the integrals over $0\leqq r\leqq 1$ of the following functions: $q_r^2(r)r^3$, $q^2(r)r^3$, $p_r^2(r)r^3$; p_r is to be considered here as a functional of q defined by the formula

$$p_r(r)=\tfrac{1}{2}r^{-3}\int_0^r q^2(r_1)r_1^3dr_1.$$

A new formulation allows a simultaneous treatment of the cases $\lambda^2<\infty$ and $\lambda^2=\infty$. The minimum problem has at most one solution. The boundary value problem will have more and more solutions as λ^2 increases; however, it has 3 solutions only ($q_1\equiv 0$, $q_3=-q_2\neq 0$) in the interval between the first and the second eigenvalues of the linearized buckling problem, which plays an important part in the bifurcation theory of the non-linear problem. An appendix is devoted to E. Schmidt's bifurcation theory. Among the mathematical devices introduced by the authors, a new application of Jacobi's transformation of the second variation is to be mentioned. This transformation has already been successfully applied by the first author in a hydrodynamical problem [see K. Friedrichs, Math. Ann. **109**, 60–82 (1933)]. *A. Weinstein* (Toronto, Ont.).

* **Jensen, Vernon P. Analyses of Skew Slabs.** University of Illinois Bulletin, v. 39, no. 3 = Engineering Experiment Station Bulletin Series no. 332. University of Illinois, Urbana, Ill., 1941. 112 pp. $1.00.

Methods of numerical solutions are given for the equation (*) $\nabla^2\nabla^2W=P$ (the equation of the deflection of a thin plate, where P is a prescribed function of x and y, w the deflection of the plate, and $\nabla^2\nabla^2$ the biharmonic operator in two dimensions) by replacing (*) by a difference equation.

Rectangular and skew nets are employed and particular attention is paid to polygonal regions. Various boundary conditions are considered. A bibliography is given of the previous engineering work on such problems.

A. E. Heins (Lafayette, Ind.).

Jeffreys, Harold. Initial stress and elastic instability. Proc. Cambridge Philos. Soc. **38**, 125–128 (1942). [MF 6039]

With a view to the application to problems of elastic instability the author establishes the fundamental relations of a theory of elasticity which admits initial stresses. The displacements from this stressed state are supposed to be small and the additional stresses are supposed to be linear functions of the infinitesimal strains. The relations to similar theories established by C. B. Biezeno and H. Hencky [Nederl. Akad. Wetensch., Proc. **31**, 569 (1928); **32**, 444 (1929)], M. Biot [Philos. Mag. (7) **27**, 468–489 (1939)] and E. Trefftz [Z. Angew. Math. Mech. **13**, 160–165 (1933)] are not discussed. *W. Prager* (Providence, R. I.).

Jeffreys, Harold. A derivation of the equations of equilibrium of a thin plate. Philos. Mag. (7) **32**, 365–368 (1941). [MF 6041]

The author shows how the treatment of thin plates given in the standard books on elasticity can be shortened to a considerable extent by the systematic use of tensor notation.

W. Prager (Providence, R. I.).

Frola, E. Instabilità elastica e generalizzazioni. (Riassunto.) Univ. e Politecnico Torino. Rend. Sem. Mat. Fis. **7**, 79–82 (1941). [MF 6279]

Tolotti, Carlo. Sul calcolo delle piastre elastiche a forma di settore anulare. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 517–525 (1940). [MF 5754]

The author transforms the problem of transverse bending of a thin elastic plate having the shape of a sector of an annulus ($r_1<r<r_2$, $0<\theta<\alpha$) by considering instead of the deflection $w(r,\theta)$ a transform $w^*(r,\lambda)=\int_0^\alpha w(r,\theta)e^{-\lambda\theta}d\theta$. From the plate equation $\nabla^2\nabla^2 w=p$ there follows then an equation $LL(w^*)=p^*+F$, where $L\equiv(d^2/dr^2+r^{-1}d/dr-\lambda^2/r^2)$ and F is a certain function of λ and of the values of w and its normal derivatives on the boundaries $\theta=0$ and α. If F did not depend on the ultimate solution of the problem itself, that solution could be obtained by determining w^* from the conditions along $r=r_1$ and r_2. As it is, F does depend on the ultimate solution, so that w has to be determined as the solution of a certain integro-differential equation. The paper contains no example of a solution obtained by this method. *E. Reissner* (Cambridge, Mass.).

Egger, Hans. Knickung der Kreisplatte und Kreisringplatte mit veränderlicher Dicke. Ing.-Arch. **12**, 190–200 (1941). [MF 5793]

The critical radial compressive load is found for annular plates for which the thickness varies as a power n of the radius, where n has the values 1/10, 1/5, 2/3, −1/5. The inner boundary is in all cases taken to be free while the outer boundary is either clamped or free. The latter case was treated by K. Federhofer [Akad. Wiss. Wien, S.-B. IIa. **149**, 59–75 (1940); these Rev. **3**, 31] who used the approximate methods of Ritz and Galerkin.

H. W. March (Madison, Wis.).

Korenev, B. G. Solution of problems of equilibrium, oscillation and stability by means of compensating loads. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 5–6, 61–72 (1940). (Russian. English summary) [MF 4652]

Application of Bessel-functions to computation of displacements in circular, triangular and square-shaped plates and membranes either in state of vibration or under action of a complicated load distribution. *M. A. Sadowsky.*

Osgood, William R. Rectangular plate loaded along two adjacent edges by couples in its own plane. J. Research Nat. Bur. Standards **28**, 159–163 (1942). [MF 6120]

The stresses in the knees of a rectangular rigid frame are studied by treating the frame as an L shaped plate and considering a two dimensional plane stress problem. By superposing the stress systems arising from two loading distributions upon a cantilever, the stresses in each leg of the frame are due to the resultant couples applied on two adjacent edges of the cantilever. Small self-equilibrated stresses remain along the free edges of the knee.

D. L. Holl (Ames, Iowa).

Ghosh, S. Stress distribution in an infinite plate containing two equal circular holes. Bull. Calcutta Math. Soc. **31**, 149–159 (1939). [MF 6149]

Expressions are given of the stresses in an infinite elastic plate which contains two equal circular holes and is subjected to (a) a uniform tension in the direction of the line joining the centers of the holes, (b) a uniform shear in the plane of the plate. *W. Prager* (Providence, R. I.).

Ohlig, R. Mehrfache prismatische Faltwerke. Ing.-Arch. **12**, 254–258 (1941). [MF 6426]

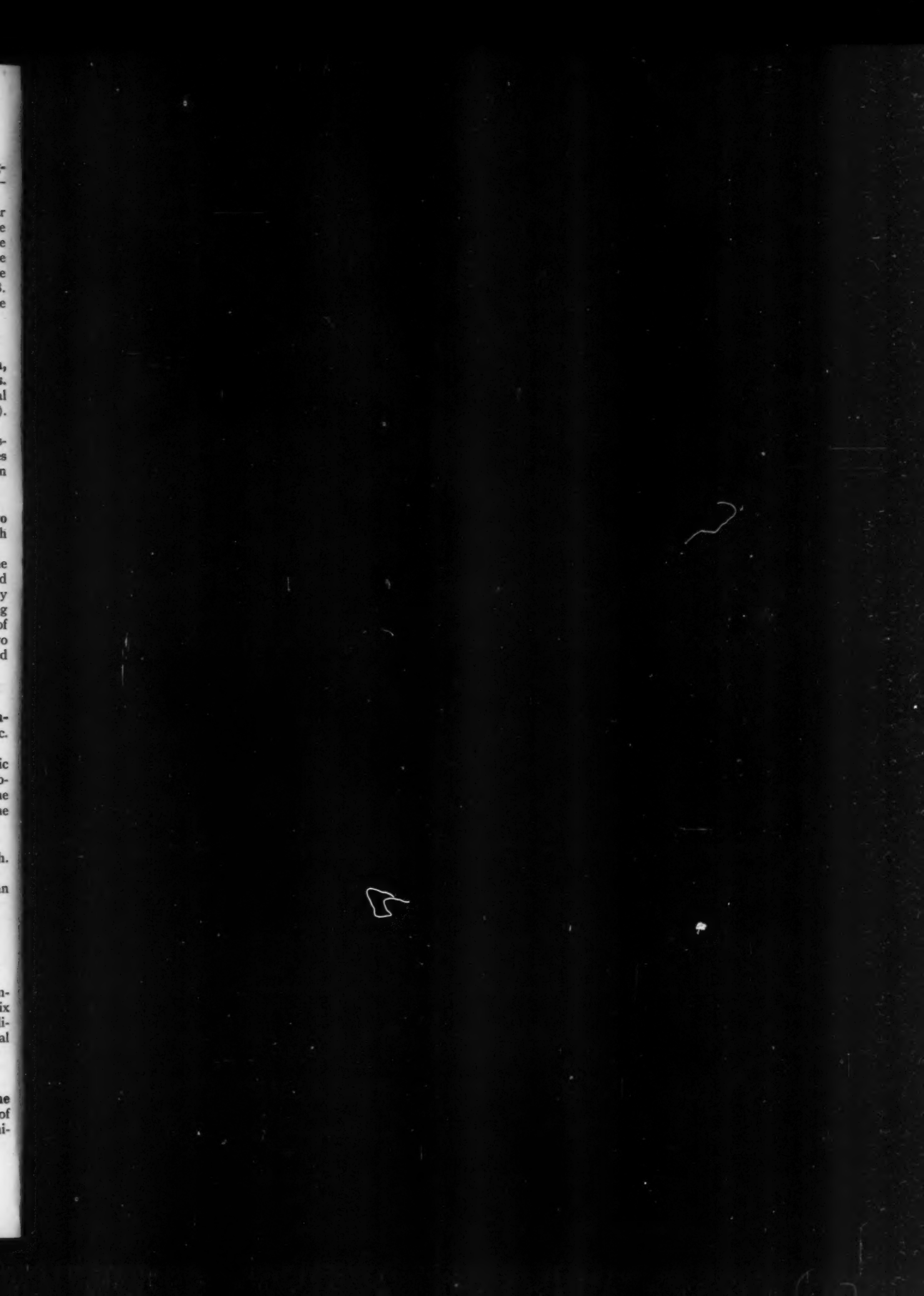
Stress analysis of paneled structures having more than two panels rigidly jointed to the edge beams.

W. Prager (Providence, R. I.).

BIBLIOGRAPHICAL NOTES

Research Publications. Illinois Institute of Technology.

According to an announcement on the second cover page: "Research Publications of Illinois Institute of Technology are issued regularly as groups of reprints selected from papers published by the teaching and research staff. Each number contains papers in a single broad field of investigation. The objective to be served by each number is to make available a group of papers related to each other and therefore of value to libraries for general reference when bound together."

Volume 1, number 1, May, 1941, bearing the subtitle Mathematics, contains 14 papers totaling 116 pages. Number 2, June, 1941, with the subtitle Mechanics, has six papers totaling 74 pages. Most of the papers contain indications of the place of original publication. The original pagination, however, has been changed.

* **Bibliography for the theory of multiple integrals in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 491–527. University of Chicago Press, Chicago, Ill., 1942. $3.00.